Study Guide 1

Solutions

1. A certificate of deposit offers an interest rate of 5.5% compounded daily for a term of 4 years. How much would you have to invest initially to have \$25000 when the CD matures?

If P_0 is invested at an interest rate r, compounded k times per year, then the value of the investment (the compound amount) after t years is given by

$$P(t) = P_0\left(1+\frac{r}{k}\right)^{kt}.$$

(See chapter 4 in the textbook for more details).

In this problem, we know $P(4)$, r and k and we want to know P_0:

$$25000 = P(4) = P_0\left(1+\frac{0.055}{365}\right)^{1460} \approx 1.246056 P_0 \implies P_0 = 25000/1.246056 \approx 20063.3.$$

2. Use the change of base formula to compute the following logarithms. Do not use a calculator — express your answers in terms of $\ln 2$, $\ln 3$, $\ln 5$, and $\ln 7$. For example

$$\log_7 100 = \frac{\ln 100}{\ln 7} = \frac{\ln 10^2}{\ln 7} = \frac{2\ln 10}{\ln 7} = \frac{2(\ln 2+\ln 5)}{\ln 7} = \frac{2\ln 2 + 2\ln 5}{\ln 7}.$$

(a) $\log_5 36 = \dfrac{\ln 36}{\ln 5} = \dfrac{2\ln 2 + 2\ln 3}{\ln 5}.$

(b) $\log_{20} 21 = \dfrac{\ln 21}{\ln 20} = \dfrac{\ln 3 + \ln 7}{2\ln 2 + \ln 5}.$

(c) $\log_{10}\sqrt{75} = \dfrac{0.5\ln 75}{\ln 10} = \dfrac{0.5\ln 3 + \ln 5}{\ln 2 + \ln 5}.$

(d) $\log_{21}\dfrac{1}{\sqrt[3]{50}} = \dfrac{-\frac{1}{3}\ln 50}{\ln 21} = -\dfrac{\frac{2}{3}\ln 5 + \frac{1}{3}\ln 2}{\ln 3 + \ln 7}.$

3. Simplify the following expressions using properties of the natural log function.

(a) $\ln\left(\dfrac{x^2+3x+1}{5x+3}\right) = \ln(x^2+3x+1) - \ln(5x+3)$

(b) $\ln\sqrt[3]{\dfrac{5xy^3}{x^2+y^2}} = \frac{1}{3}\left(\ln 5 + \ln x + 3\ln y - \ln\left(x^2+y^2\right)\right)$

$$= \tfrac{1}{3}\ln 5 + \tfrac{1}{3}\ln x + \ln y - \tfrac{1}{3}\ln(x^2+y^2).$$

4. Solve the equations.

(a) $3x^2+5x-8=0 \implies x=\dfrac{-5\pm\sqrt{25+96}}{6}=\dfrac{-5\pm 11}{6} \implies \begin{cases} x_1 &= 1 \\ x_2 &= -8/3 \end{cases}$

(b) $\dfrac{2x+1}{x-2}=\dfrac{3x+5}{8-2x} \implies (2x+1)(8-2x)=(3x+5)(x-2) \implies 7x^2-15x-18=0$

$\implies x=\dfrac{15\pm\sqrt{225+504}}{14}=\dfrac{15\pm 27}{14} \implies \begin{cases} x_1 &= 3 \\ x_2 &= -6/7 \end{cases}$

5. Solve the pairs of equations.

(a) $\begin{cases} 4x+5y &= 7 \\ 3x+4y &= 13 \end{cases}$

First approach:

(i) Subtract $5\times(3x+4y=13)$ from $4\times(4x+5y=7)$:

$$\begin{array}{rcr} 4\times(4x+5y & = & 7) \\ -\ 5\times(3x+4y & = & 13) \\ \hline x & = & -37 \end{array}$$

(ii) Subtract $3\times(4x+5y=7)$ from $4\times(3x+4y=13)$:

$$\begin{array}{rcr} 4\times(3x+4y & = & 13) \\ -\ 3\times(4x+5y & = & 7) \\ \hline y & = & 31 \end{array}$$

(iii) Solution: $(x,y)=(-37,31)$.

Second approach:

(i) Solve $4x+5y=7$ for y (or for x): $y=\dfrac{7-4x}{5}$.

(ii) Substitute this into the second equation:

$$3x+4y=13 \implies 3x+4\left(\frac{7-4x}{5}\right)=13 \implies 15x+28-6x=65 \implies x=-37,$$

(iii) Find the corresponding y-value:

$$y=\frac{7-4(-37)}{5}=\frac{7+148}{5}=\frac{155}{5}=31.$$

(iv) Solution (as before): $(x,y)=(-37,31)$.

(b) $\begin{cases} 3x-2y &= 1 \\ 5x+y &= 2 \end{cases}$

I'll use the first approach for this one:

(i) Add $(3x-2y=1)$ to $2\times(5x+y=2)$:

$$\begin{array}{rcr} (3x-2y & = & 1) \\ +2\ (5x+y & = & 2) \\ \hline 13x & = & 5 \end{array} \implies x=\frac{5}{13}$$

(ii) Subtract $3 \times (5x + y = 2)$ from $5 \times (3x - 2y = 1)$:

$$\begin{array}{rcl} 5 \times (3x - 2y & = & 1) \\ -\ 3\ \times (5x + y & = & 2) \\ \hline -13y & = & -1 \end{array} \implies y = \frac{1}{13}$$

(iii) Solution: $(x, y) = (5/13, 1/13)$.

(c) $\begin{cases} x^2 + 2x - 3y & = & -1 \\ 4x + 2y & = & 14 \end{cases}$

I'll use the second approach here:

(i) Solve $4x + 2y = 14$ for y: $y = 7 - 2x$.

(ii) Substitute this into the first equation:

$$x^2 + 2x - 3y = -1 \implies x^2 + 2x - 3(7 - 2x) = -1 \implies x^2 + 8x - 20 = 0$$

$$\implies x = \frac{-8 \pm \sqrt{64 + 80}}{2} = \frac{-8 \pm 12}{2} \implies \begin{cases} x_1 & = & 2 \\ x_2 & = & -10 \end{cases}$$

(iii) Find the corresponding y-values:

$$y_1 = 7 - 2x_1 = 3 \text{ and } y_2 = 7 - 2x_2 = 27.$$

(iv) Solutions: $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-10, 27)$.

6. Compute the following limits.

(a) $\lim_{x \to 3} \frac{x^3 - 8}{x - 2} = \frac{\lim_{x \to 3} x^3 - 8}{\lim_{x \to 3} x - 2} = \frac{19}{1} = 19.$

*The limit of a quotient equals the quotient of the limits **if** (both limits exist and) the limit of the denominator is not* 0 .

(b) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x - 2)}(x^2 + 2x + 4)}{\cancel{x - 2}} = \lim_{x \to 2} (x^2 + 2x + 4) = 4 + 4 + 4 = 12.$

*In this case, the limit of the original denominator **is*** 0*, but I was able to factor the numerator and cancel out the denominator (before computing the limit).*

(c) $\lim_{x \to \infty} \frac{x^3 - 8}{x - 2} = \lim_{x \to \infty} \frac{x^3}{x} = \lim_{x \to \infty} x^2 = \infty$

The limit 'at infinity' of a rational function is equal to the limit at infinity of the quotient of the highest order terms in the numerator and denominator, respectively.

(d) $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h} = \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x$

As in (b), the limit of the denominator is 0, so we have to simplify and cancel the denominator before calculating the limit.

(e) $\displaystyle\lim_{y\to\infty} \frac{5y^4 + 300y^2 - 60y + 1000}{20000 - 3y + 4y^3 - 0.01y^4} = \lim_{y\to\infty} \frac{5\cancel{y^4}}{-0.01\cancel{y^4}} = \lim_{y\to\infty} -500 = -500$

This one is similar to (c).

(f) $$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{\left(\sqrt{4+x}-2\right)\left(\sqrt{4+x}+2\right)}{x\left(\sqrt{4+x}+2\right)} = \lim_{x\to 0} \frac{\left(\sqrt{4+x}\right)^2 - 2^2}{x\left(\sqrt{4+x}+2\right)} = \lim_{x\to 0} \frac{\cancel{4}+x\cancel{-4}}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{4+x}+2\right)} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\left(\sqrt{4+x}+2\right)} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}.$$

Sometimes, simplification requires an initial multiplication by the same factor, top and bottom.

(g) $\displaystyle\lim_{t\to 2^+} \frac{t-2}{\sqrt{t-2}} = \lim_{t\to 2^+} \frac{\cancel{\left(\sqrt{t-2}\right)}\left(\sqrt{t-2}\right)}{\cancel{\sqrt{t-2}}} = \lim_{t\to 2^+} \sqrt{t-2} = \sqrt{2-2} = 0.$

A one-sided limit is required here because $\sqrt{t-2}$ is not defined when $t < 2$.

(h) $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{\cancel{x}}{-\cancel{x}} = \lim_{x\to 0^-} \frac{1}{-1} = -1.$

If $x < 0$, as it is when $x \to 0^-$, then $|x| = -x$.

(i) $\displaystyle\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{\cancel{x}}{\cancel{x}} = \lim_{x\to 0^-} \frac{1}{1} = 1.$

If $x > 0$, as it is when $x \to 0^+$, then $|x| = x$.