Study Guide 2

Solutions

1. Without consulting your notes or book, give the mathematical definitions for the two bold-faced concepts below. You should use *limits* in both definitions.

- The function $f(x)$ is **continuous** at the point $x = a$.
- The function $f(x)$ is **differentiable** at the point $x = a$.

The definitions:

- The function $f(x)$ is **continuous** at the point $x = a$ *if and only if* $\lim_{x \to a} f(x)$ *exists and*
$$\lim_{x \to a} f(x) = f(a).$$
- The function $f(x)$ is **differentiable** at the point $x = a$ *if and only if*
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
exists.

By fiddling with the limits in these two definitions, we can emphasize the similarity as well as the difference between these two concepts. Specifically, $f(x)$ is **continuous** at $x = a$ if and only if
$$\lim_{x \to a} f(x) - f(a) = 0,$$
and $f(x)$ is **differentiable** at $x = a$ if and only if
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists.}$$

Question: Explain how you go from the first pair of limits defining continuity and differentiability to the second pair.

2. Find the points of **discontinuity** of the function $f(x) = \dfrac{x^2+1}{x^2-4x+3}$, and use one-sided limits to describe how the function behaves near these points.

The function $f(x) = \dfrac{x^2+1}{x^2-4x+3}$ is continuous for all x except where the denominator of $f(x)$ is equal to 0. In other words, the points of **discontinuity** of $f(x)$ are the points where $x^2 - 4x + 3 = 0$. The quadratic expression on the left of the equation is easy to factor† in this case
$$x^2 - 4x + 3 = 0 \implies (x-1)(x-3) = 0 \implies x = 1 \text{ or } x = 3,$$
so the points of discontinuity of $f(x)$ are $x_1 = 1$ and $x_2 = 3$.

Next, to determine the behavior of $f(x)$ near the points of discontinuity we compute both one-sided limits of $f(x)$ at each point. First, a couple of observations

†Factoring is fine when it works but in general, I **strongly** recommend relying on the **quadratic formula** to solve quadratic equations. See SN2 on the review page.

(i) The numerator of $f(x)$ is **always positive**, in fact $x^2+1 \geq 1$ for all x. This means that all of the one-sided limits below will be infinite, either $+\infty$ or $-\infty$, depending on whether $f(x)$ is positive or negative as x approaches the limiting point, which in turn will depend on **the side** of the limiting point from which x is approaching.

(ii) Because of the fact that the numerator of $f(x)$ is always positive, it follows that $f(x)$ always has the same sign as its denominator. Now, from the factorization $x^2-4x+3 = (x-1)(x-3)$ we conclude that the denominator of $f(x)$ is positive when $x < 1$; is negative when $1 < x < 3$; positive when $3 < x$.

On to the limits ...

(*) $\displaystyle\lim_{x\to 1^-} \frac{x^2+1}{x^2-4x+3} = +\infty$, because

(a) the numerator is approaching 2 and the denominator is approaching 0, and

(b) $f(x)$ is positive when $x < 1$.

(*) $\displaystyle\lim_{x\to 1^+} \frac{x^2+1}{x^2-4x+3} = -\infty$, because

(a) the numerator is approaching 2 and the denominator is approaching 0, and

(b) $f(x)$ is negative when $1 < x < 3$.

(*) $\displaystyle\lim_{x\to 3^-} \frac{x^2+1}{x^2-4x+3} = -\infty$, because

(a) the numerator is approaching 10 and the denominator is approaching 0, and

(b) $f(x)$ is negative when $1 < x < 3$.

(*) $\displaystyle\lim_{x\to 3^+} \frac{x^2+1}{x^2-4x+3} = +\infty$, because

(a) the numerator is approaching 10 and the denominator is approaching 0, and

(b) $f(x)$ is positive when $3 < x$.

The behavior of $f(x)$ around its points of discontinuity is illustrated in Figure 1, below.

3. What can you say about $\displaystyle\lim_{x\to 0} \frac{x}{|x|}$?

This limit does not exist because the two one-sided limits $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|} = -1$ and $\displaystyle\lim_{x\to 0^+} \frac{x}{|x|} = 1$ are ***not*** equal (though they both exist). See problems 6(h) and 6(i) of Study Guide 1.

4. Give an example of a function that is **continuous** at the point $x = 0$, but is **not differentiable** at $x = 0$. Justify your answer.

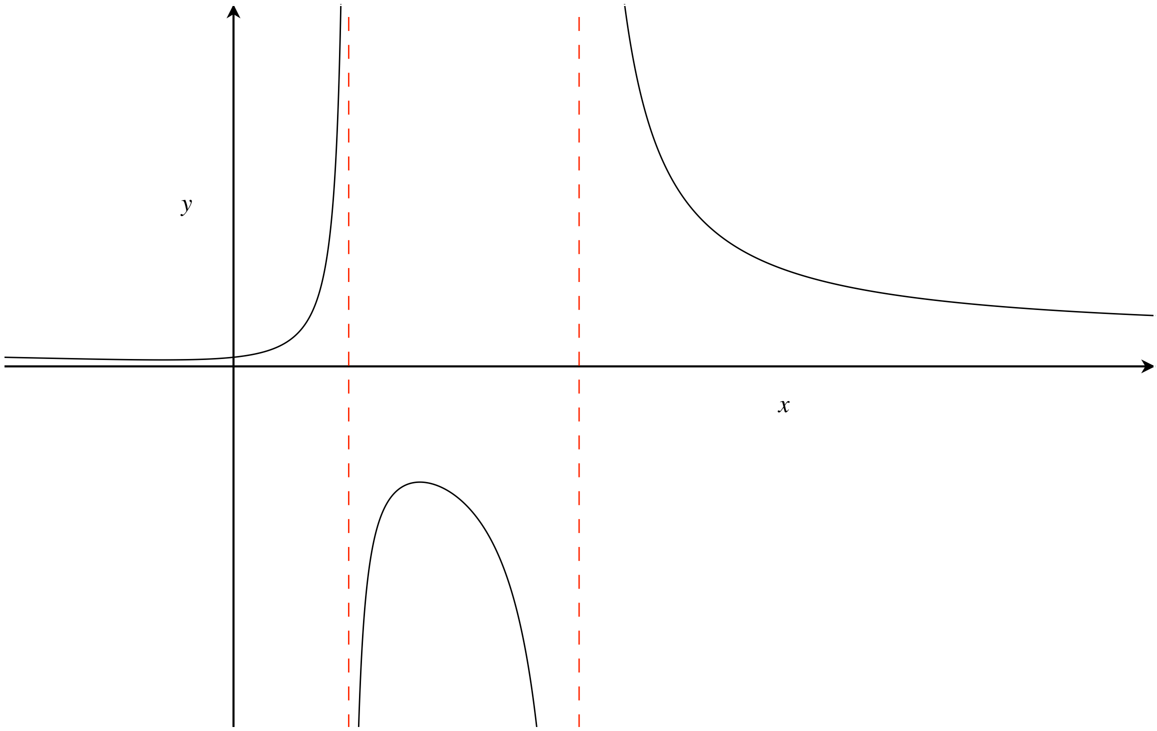

Figure 1: The graph of $f(x) = \dfrac{x^2+1}{x^2-4x+3}$

The function $f(x) = |x|$ is continuous at $x = 0$, because

$$\lim_{x\to 0} |x| = \left\{ \begin{array}{lcl} \lim\limits_{x\to 0^+} x & : & x > 0 \\ \lim\limits_{x\to 0^-} -x & : & x < 0 \end{array} \right\} = 0 = |0|.$$

On the other hand, $\dfrac{f(0+h)-f(0)}{h} = \dfrac{|h|}{h}$, so

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{\cancel{h}}{\cancel{h}} = \lim_{h\to 0^+} 1 = 1$$

but

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-\cancel{h}}{\cancel{h}} = \lim_{h\to 0^+} -1 = -1.$$

Since the two one-sided limits are not equal, the limit $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ does not exist, so $f(x) = |x|$ is not differentiable at $x = 0$.

5. For each of the functions $f(x)$ given below, simplify the expression $\dfrac{f(x+h)-f(x)}{h}$. Express your answer in terms of x and h.

(a) $f(x) = x^2$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = 2x + h.$$

(b) $f(x) = x^3$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \frac{3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}}}{\cancel{h}} = 3x^2 + 3xh + h^2$$

(c) $f(x) = \sqrt{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

(d) $f(x) = \dfrac{1}{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \frac{\cancel{x - x} - h}{hx(x+h)}$$

$$= \frac{-\cancel{h}}{\cancel{h}x(x+h)} = -\frac{1}{x(x+h)}$$

6. For the functions and points given below, *use the definition of the derivative* to find the slope of the graph of $y = f(x)$ at the point x_0, and the equation of the tangent line to the graph $y = f(x)$ at the point $(x_0, f(x_0))$.

(a) $f(x) = x^2; \quad x_0 = 1.$

(i) The slope:

$$f'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{1^2 + 2h + h^2 - 1}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} 2 + h = 2$$

(ii) Equation of the tangent line:

$$y - f(1) = f'(1)(x-1) \implies y - 1 = 2(x-1) \implies y = 2(x-1) + 1 \implies \boxed{y = 2x - 1}.$$

(b) $f(x) = 3x^2 + 2x - 1; \quad x_0 = 2.$

(i) The slope:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{3(2+h)^2 + 2(2+h) - 1 - (3\cdot 2^2 + 2\cdot 2 - 1)}{h} \\ &= \lim_{h\to 0} \frac{12 + 3\cdot 4h + 3h^2 + 4 + 2h - 1 - 12 - 4 + 1}{h} \\ &= \lim_{h\to 0} \frac{12h + 3h^2 + 2h}{h} = \lim_{h\to 0} 12 + 3h + 2 = 14. \end{aligned}$$

(ii) Equation of the tangent line:

$$y - f(2) = f'(2)(x-2) \implies y - 15 = 14(x-2) \implies y = 14(x-2) + 15 \implies \boxed{y = 14x - 13}.$$

(c) $f(x) = \dfrac{2}{x}; \quad x_0 = 3.$

(i) The slope:

$$f'(3) = \lim_{h\to 0} \frac{\dfrac{2}{3+h} - \dfrac{2}{3}}{h} = \lim_{h\to 0} \frac{\dfrac{6}{3(3+h)} - \dfrac{2(3+h)}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{6} - \cancel{6} - 2h}{3h(3+h)} = \lim_{h\to 0} \frac{-2\cancel{h}}{3\cancel{h}(3+h)} = \lim_{h\to 0} \frac{-2}{3(3+h)} = -\frac{2}{9}$$

(ii) Equation of the tangent line:

$$y - f(3) = f'(3)(x-3) \implies y - \frac{2}{3} = -\frac{2}{9}(x-3) \implies y = -\frac{2}{9}(x-3) + \frac{2}{3} \implies \boxed{y = -\frac{2}{9}x + \frac{4}{3}}.$$

(d) $f(x) = 3\sqrt{x}; \quad x_0 = 4.$

(i) The slope:

$$f'(4) = \lim_{h\to 0} \frac{3\sqrt{4+h} - 3\sqrt{4}}{h} = \lim_{h\to 0} \frac{3\sqrt{4+h} - 6}{h} = \lim_{h\to 0} \frac{\left(3\sqrt{4+h} - 6\right)\left(3\sqrt{4+h} + 6\right)}{h\left(3\sqrt{4+h} + 6\right)}$$

$$= \lim_{h\to 0} \frac{(9(4+h) - 36)}{h\left(3\sqrt{4+h} + 6\right)} = \lim_{h\to 0} \frac{\cancel{36} + 9h - \cancel{36}}{h\left(3\sqrt{4+h} + 6\right)} = \lim_{h\to 0} \frac{9\cancel{h}}{\cancel{h}\left(3\sqrt{4+h} + 6\right)}$$

$$= \lim_{h\to 0} \frac{9}{3\sqrt{4+h} + 6} = \frac{9}{12} = \frac{3}{4}$$

(ii) Equation of the tangent line:

$$y - f(4) = f'(4)(x-4) \implies y - 6 = \frac{3}{4}(x-4) \implies y = \frac{3}{4}(x-4) + 6 \implies \boxed{y = \frac{3}{4}x + 3}.$$

7. Use the ***rules of differentiation*** to compute the derivatives of the functions below.

(a) $f(x) = 3x^2 - 2x + 5$

$f'(x) = 3 \cdot (2x) - 2 \cdot 1 = 6x - 2$

(b) $y = 2\sqrt{x} - \dfrac{3}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^{1/2} - 3x^{-2}\right) = 2 \cdot \left(\tfrac{1}{2}x^{-1/2}\right) - 3 \cdot \left(-2x^{-3}\right) = x^{-1/2} + 6x^{-3} \qquad \left(= \frac{1}{\sqrt{x}} + \frac{6}{x^3}\right)$$

(c) $g(t) = \dfrac{3t+4}{t^2+5}$

$$g'(t) = \frac{3(t^2+5) - 2t(3t+4)}{(t^2+5)^2} = \frac{-3t^2 - 8t + 15}{(t^2+5)^2} \qquad \text{(Quotient Rule)}$$

(d) $h(x) = (2x+5)(x^2+2x+3)$

$h'(x) = 2(x^2+2x+3) + (2x+5)(2x+2) = 6x^2 + 18x + 16 \qquad$ (Product Rule)

8. (a) Find the derivative of $f(x) = \sqrt[3]{x}$ at the point $x = 8$.

$$f(x) = \sqrt[3]{x} = x^{1/3} \implies f'(x) = \frac{1}{3}x^{-2/3} \implies f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12}$$

(b) Use your answer to (a) and *linear approximation* to estimate $\sqrt[3]{9}$.

Linear approximation: *if x_1 is close to x_0,*[‡] *then*

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0).$$

In this case $x_0 = 8$ and $x_1 = 9$, so

$$\sqrt[3]{9} = f(9) \approx f(8) + f'(8)(9-8) = 2 + \frac{1}{12} \cdot 1 = \frac{25}{12} = 2.08333\ldots.$$

The calculator estimate is $\sqrt[3]{9} \approx 2.0800838\ldots$, so the linear approximation above is off by less than 0.0033.

(c) Use the same ideas to estimate $\sqrt{102}$.

Use linear approximation for the function $f(x) = \sqrt{x} = x^{1/2}$ and the points $x_0 = 100$ and $x_1 = 102$.[§] With these choices, we have

$$f'(x) = \frac{1}{2}x^{-1/2} \implies f'(100) = \frac{1}{20},$$

and therefore

$$\sqrt{102} = f(102) \approx f(100) + f'(100)(102 - 100) = 10 + \frac{1}{20} \cdot 2 = 10.1.$$

The calculator estimate is $\sqrt{100} \approx 10.0995049\ldots$, so the linear approximation above is off by less than 0.0005.

9. A firm's marginal revenue function is given by

$$\frac{dr}{dq} = 0.7q - 0.05q^2,$$

where revenue r is measured in \$1000s and output q is measured in 100s of units. By approximately how much will the firm's revenue change if output increases from 1000 units to 1050 units?

Note that we don't know the revenue function in this case, so we can't compute the revenue directly at any point,[¶] nonetheless, we can estimate the *change* in the revenue using linear approximation. Specifically, we use linear approximation in this form:

$$r(q_1) - r(q_0) \approx \left(\left.\frac{dr}{dq}\right|_{q=q_0} \right)(q_1 - q_0).$$

[‡]There is no uniform definition for 'close'. You may assume that the points are close enough if you are asked to use linear approximation. There are methods for estimating the size of the error of approximation, depending on the size of $|x_1 - x_0|$, but we won't go into that at this point.

[§]The idea is to find a point x_0 such that (a) x_0 is close to the point x_1 ($x_1 = 102$ in this case), and (b) $f(x)$ and $f'(x)$ are relatively easy to evaluate at x_0.

[¶]We will learn how to do this in 11B.

In this problem, $q_0 = 10$ and $q_1 = 10.5$, because we are measuring output in 100s of units, and therefore

$$\left.\frac{dr}{dq}\right|_{q=10} = 7 - 5 = 2$$

so

$$r(10.5) - r(10) \approx 2 \cdot 0.5 = 1,$$

which means that the firm's revenue will increase by about \$1000.