Solutions

1. Compute the derivatives of the functions below

a. $f(x) = \sqrt[3]{x^3 - 3x^2 + 1}$ **d.** $g(x) = \log_5 x$ **g.** $y = \ln(x^2 + 2x + 3)$

b. $y = \ln(5x + 3)$ **e.** $h(u) = 3^u$ **h.** $g(t) = e^{t^3 - 2t + 1}$

c. $s = e^{0.05t}$ **f.** $f(x) = 3x^2 e^x$ **i.** $k(u) = \dfrac{u \ln u}{2u + 1}$

The derivatives....

a. $f'(x) = \frac{1}{3}(x^3 - 3x^2 + 1)^{-2/3} \cdot (3x^2 - 6x) = \dfrac{x^2 - 2x}{(x^3 - 3x^2 + 1)^{2/3}}$. *power rule.*

b. $y' = \dfrac{5}{5x + 3}$. *chain rule.*

c. $\dfrac{ds}{dt} = (0.05)e^{0.05t}$. *chain rule*

d. $g'(x) = \dfrac{1}{\ln 5} \cdot \dfrac{1}{x}$. $\log_5 x = \dfrac{\ln x}{\ln 5}$.

e. $h'(u) = (\ln 3)3^u$. $3^u = e^{(\ln 3)u}$, now use *chain rule.*

f. $f'(x) = 6xe^x + 3x^2 e^x = 3(2x + x^2)e^x$. *Product rule.*

g. $y' = \dfrac{2x + 2}{x^2 + 2x + 3}$. *chain rule.*

h. $g'(t) = (3t^2 - 2)e^{t^3 - 2t + 1}$. *chain rule.*

i. $k'(u) = \dfrac{(1 \cdot \ln u + u \cdot \frac{1}{u})(2u + 1) - 2(u \ln u)}{(2u + 1)^2}$ *Quotient rule* and *product rule*

$= \dfrac{(\ln u + 1)(2u + 1) - 2u \ln u}{(2u + 1)^2}$ for the numerator.

$= \dfrac{2u + \ln u + 1}{(2u + 1)^2}$ Then clean up.

2. The consumption function for a small country is given by

$$C = \ln\left(\frac{e^{0.95Y}}{e^{0.2Y} + 5}\right),$$

where Y is national income, measured in \$ billions.

a. When $Y = 10$ consumption is ...

$C(10) = \ln\left(\dfrac{e^{9.5}}{e^2 + 5}\right) \approx 6.983$, or about 6 billion 983 million dollars.

b. First note that $C = \ln e^{0.95Y} - \ln\left(e^{0.2Y} + 5\right) = 0.95Y - \ln\left(e^{0.2Y} + 5\right)$.

So $\dfrac{dC}{dY} = 0.95 - \dfrac{0.2e^{0.2Y}}{e^{0.2Y}+5}$, and $\left.\dfrac{dC}{dY}\right|_{Y=10} = 0.95 - \dfrac{0.2e^2}{e^2+5} \approx 0.831.$

c. By approximately how much will *consumption and savings* increase if income increases from \$10 billion to \$10.4 billion?

First consumption, using linear approximation:

$$\Delta C \approx \left.\frac{dC}{dY}\right|_{Y=10} \cdot \Delta Y \approx 0.831 \cdot 0.4 = 0.3324$$

I.e., if income increases from \$10 billion to \$10.4 billion, consumption will increase by about $\$332,400,000$.

Next, savings, Recall that $\dfrac{dS}{dY} = 1 - \dfrac{dC}{dY}$, so $\left.\dfrac{dS}{dY}\right|_{Y=10} \approx 1 - 0.831 = 0.169$. Therefore, by the linear approximation again

$$\Delta S \approx \left.\frac{dS}{dY}\right|_{Y=10} \cdot \Delta Y \implies \Delta S \approx 0.169 \cdot 0.4 = 0.0676.$$

Alternatively, it is also true that $\Delta Y = \Delta S + \Delta C$, so

$$\Delta S = \Delta Y - \Delta C \approx 0.4 - 0.3324 = 0.0676.$$

Either way, if income increases from \$10 billion to \$10.4 billion, savings will increase by about \$67,600,000.00

d. $$\begin{aligned}\lim_{Y\to\infty}\frac{dC}{dY} &= \lim_{Y\to\infty}\left(0.95 - \frac{0.2e^{0.2Y}}{e^{0.2Y}+5}\right)\\ &= \lim_{Y\to\infty} 0.95 - \lim_{Y\to\infty}\left(\frac{0.2e^{0.2Y}}{e^{0.2Y}+5}\right)\\ &= 0.95 - \lim_{Y\to\infty}\left(\frac{0.2}{1+5e^{-0.2Y}}\right) \qquad \text{(Divide numerator and denominator by } e^{0.2Y}.\text{)}\\ &= 0.95 - \frac{0.2}{1} = 0.75. \qquad \text{(Since } \lim_{Y\to\infty} e^{-0.2Y} = 0.\text{)}\end{aligned}$$

Interpretation: When income grows sufficiently large, the MPC will stabilize at about 0.75, i.e., when income is large, the nation will consume about 75 cents of each additional dollar of income (and save the other 25 cents).

3. The *marginal revenue* function of a monopolistic firm is given by

$$\frac{dr}{dq} = \sqrt{250-q},$$

where revenue is measured in \$1000s per month and the firm's output q is measured in 1000s of units per month. The firm's production function is

$$q = 30(4m-15)^{1/3},$$

where m is the firm's labor input measured in 40-hour work weeks (e.g., if $m = 5$, then the firm's employees are working a combined 200 hours a week and if $m = 17.2$, then the firm's employees are working a combined 4288 hours a week). The firm's current labor input is $m = 35$.

a. Find the firm's output and *marginal product of labor* at the current level of labor input.

When labor input is $m = 35$, output is $q(35) = 30(4 \cdot 35 - 15)^{1/3} = 150$ and

$$\frac{dq}{dm} = 30 \cdot \frac{1}{3}(4m - 15)^{-2/3} \cdot 4 = 40(4m - 15)^{-2/3} \implies \left.\frac{dq}{dm}\right|_{m=35} = 40 \cdot 125^{-2/3} = 1.6.$$

b. Find the firm's *marginal revenue product* at the current level of labor input.

Marginal revenue product is dr/dm, and by the chain rule

$$\left.\frac{dr}{dm}\right|_{m=35} = \left.\frac{dr}{dq}\right|_{m=35} \cdot \left.\frac{dq}{dm}\right|_{m=35} = \left.\frac{dr}{dq}\right|_{q=150} \cdot \left.\frac{dq}{dm}\right|_{m=35} = \sqrt{100} \cdot 1.6 = 16.$$

c. The firm decides to increase its labor force and hires an additional part-time laborer, to work 10 hours a week. The total monthly expense to the firm for the new employee is \$2450.00. Use your answer to part **b.** to estimate the change in the firm's monthly profit.

If the firm increases its labor input by $\Delta m = \frac{10}{40} = 0.25$, then the change to the firm's revenue is

$$\Delta r \approx \left.\frac{dr}{dm}\right|_{m=35} \cdot \Delta m = 16 \cdot 0.25 = 4.$$

Revenue is measured in \$1000s, so the firm's revenue will increase by about \$4000.00, so its profit will increase by about \$4000.00-\$2450.00=\$1550.00.

4. Find the *labor-elasticity of output* for the firm in the problem above when $m = 35$. Use your answer to estimate the *percentage* change in output, if the firm increases its labor input by 30 hours a week.

The labor-elasticity of output is given by

$$\eta_{q/m} = \frac{dq}{dm} \cdot \frac{m}{q}.$$

When $m = 35$, we know that $q = 150$ and $dq/dm = 1.6 = 8/5$, so

$$\left.\eta_{q/m}\right|_{m=35} = \frac{8}{5} \cdot \frac{35}{150} = \frac{28}{75}.$$

To estimate the percentage change in output, we use the approximation

$$\%\Delta q \approx \eta_{q/m} \cdot \%\Delta m.$$

The percentage change in labor input is

$$\%\Delta m = \frac{\Delta m}{m} \cdot 100\% = \frac{0.75}{35} \cdot 100\% = \frac{75}{35}\%$$

so the percentage change in output will be

$$\%\Delta q \approx \frac{28}{75} \cdot \frac{75}{35}\% = 0.8\%$$

Comment: If we simply evaluate the production function at $m_1 = 35.75$ we get

$$q(35.75) \approx 151.1905 \implies \%\Delta q = \frac{\Delta q}{q} \cdot 100\% \approx \frac{1.1905}{15} \cdot 100\% \approx 0.793\%$$

so (linear) approximation worked well here.

5. The demand equation for a monopolist's product is $p = 250 - 0.2q$.

a. Find the price-elasticity of demand (as a function of q).

$$\eta_{q/p} = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{250-0.2q}{q}}{-0.2} = \frac{250-0.2q}{-0.2q} = \frac{q-1250}{q} = 1 - \frac{1250}{q}.$$

b. What is the price elasticity of demand when $p = \$50$? Is demand elastic, inelastic, or does demand have unit elasticity at this point?

To answer this, we first need to find the value of q when $p = 50$:

$$p = 50 \implies 50 = 250 - 0.2q \implies 0.2q = 200 \implies q = 1000.$$

Now, compute $\eta = 1 - \frac{1250}{1000} = -0.25$. Since $|\eta| < 1$, it follows that demand is *inelastic* at that point on the demand curve.

c. Suppose that the price is lowered (from \$50) to \$49.25. Use your answer to part b. to estimate the *percentage* change in demand.

We use the approximation, $\%\Delta q \approx \eta \cdot (\%\Delta p)$ and first we need to compute the percentage change in price:

$$\%\Delta p = \frac{p_{\text{new}} - p_{\text{old}}}{p_{\text{old}}} \cdot 100\% = \frac{49.25 - 50}{50} \cdot 100\% = -1.5\%.$$

Now, we can compute the (approximate) percentage change in demand,

$$\%\Delta q \approx (-0.25)(-1.5\%) = 0.375\%.$$

I.e., demand will *increase* by approximately $\frac{3}{8}$ of 1 percent.

d. What effect will this change in price have on the firm's revenue? Be as precise as you can, and explain your answer.

To give the most precise answer, we utilize the relationship between elasticity and marginal revenue:

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right).$$

In the example above, when $q = 1000$ and $p = 50$, $\eta = -0.25$, so

$$\frac{dr}{dq} = 50(1 - 1/0.25) = 50(1-4) = -150 < 0.$$

Since marginal revenue is negative when $q = 1000$, an increase in q brings about a *decrease* in revenue, and since lowering the price raises demand, the effect on revenue of lowering the price will be to lower revenue.

6. Find the cubic Taylor polynomial for the function $f(x) = \sqrt{x}$, centered at the point $x_0 = 25$. Use this polynomial to estimate $\sqrt{26}$.

To find the cubic Taylor polynomial for $f(x) = x^{1/2}$, we need to find $f(25)$, $f'(25)$, $f''(25)$ and $f'''(25)$, so first, we need to differentiate three times:

$$f'(x) = \frac{1}{2}x^{-1/2}, \ f''(x) = -\frac{1}{4}x^{-3/2} \text{ and } f'''(x) = \frac{3}{8}x^{-5/2}.$$

Next we evaluate:

$$f(25) = 5, \ f'(25) = \frac{1}{10}, \ f''(25) = -\frac{1}{500} \text{ and } f'''(25) = \frac{3}{25000}.$$

Finally, we put these numbers into the formula for the cubic Taylor polynomial

$$T_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{6}(x - x_0)^3,$$

which in this case gives

$$T_3(x) = 5 + \frac{1}{10}(x - 25) - \frac{1}{1,000}(x - 25)^2 + \frac{1}{50,000}(x - 25)^3.$$

Finally, using $T_3(26)$ to approximate $\sqrt{26}$, we find that

$$\sqrt{26} = f(26) \approx T_3(26) = 5 + \frac{1}{10} - \frac{1}{1,000} + \frac{1}{50,000} = 5.09902.$$

My calculator says that $\sqrt{26} = 5.09901951\ldots$, so the error of approximation in this case is less than 0.0000005.