Optimization

1. Find the critical points and critical values of the following functions.

a. $f(x) = 3x^2 - 4x + 2$

Critical point:

$$f'(x) = 0 \implies 6x - 4 = 0 \implies x_1 = \frac{2}{3}$$

Critical value $f(x_1) = f(2/3) = 2/3$.

b. $g(t) = 2t^3 - 9t^2 - 24t + 7$

Critical points:

$$g'(t) = 0 \implies 6t^2 - 18t - 24 = 0 \implies 6(t-4)(t+1) = 0 \implies \begin{cases} t_1 &= 4 \\ t_2 &= -1 \end{cases}$$

Critical values: $g(t_1) = g(4) = -105$ and $g(t_2) = g(-1) = 20$.

c. $y = 5xe^{-0.125x^2}$

Critical point(s):

$$y' = 0 \implies 5e^{-0.125x^2} - 1.25x^2e^{-0.125x^2} = 0 \implies e^{-0.125x^2}(5 - 1.25x^2) = 0$$

Recall that $e^u > 0$ for all u, and therefore

$$y' = 0 \implies 5 - 1.25x^2 = 0 \implies x^2 = 4 \implies \begin{cases} x_1 &= 2 \\ x_2 &= -2 \end{cases}$$

Critical values: $y(x_1) = y(2) = 10e^{-0.5}$ (≈ 6.0653) and $y(x_2) = y(-2) = -10e^{-0.5}$.

d. $w = \dfrac{9u}{4+5u} - u$

Critical points:

$$\frac{dw}{du} = 0 \implies \frac{36}{(4+5u)^2} - 1 = 0 \implies 36 = (4+5u)^2 \implies 4+5u = \pm 6 \implies \begin{cases} u_1 &= 0.4 \\ u_2 &= -2 \end{cases}$$

Critical values: $w(u_1) = w(0.4) = 0.2$ and $w(u_2) = w(-2) = 5$

2. Use the ***first derivative test*** to classify the critical values that you found in 1c. and 1d. as relative minimum values, relative maximum values or neither.

1c. We check the ***sign*** of the derivative at a point to the left of $x_2 = -2$, at a point between the two critical points and to the right of $x_2 = 2$. First:

$$y'(-3) = e^{-9/8}(5 - 11.25) = -6.25e^{-9/8} < 0 \text{ and } y'(0) = 5 > 0$$

so y is decreasing as we approach $x_2 = -2$ from the left and y is increasing as we leave $x_2 = -2$ on the right, and therefore $y(-2) = -10e^{-0.5}$ is a relative *minimum* value. Next:

$$y'(0) = 5 > 0 \text{ and } y'(2) = e^{-9/8}(5 - 11.25) = -6.25e^{-9/8} < 0$$

so y is increasing as we approach $x_= 2$ from the left and y is decreasing as we leave $x_1 = 2$ on the right, and therefore $y(2) = 10e^{-0.5}$ is a relative *maximum* value.

1d. Same process here...

$$\left.\frac{dw}{du}\right|_{u=-3} = \frac{36}{121} - 1 < 0 \ \text{ and } \ \left.\frac{dw}{du}\right|_{u=0} = 8 > 0$$

so w is decreasing as we approach $u_2 = -2$ from the left and w is increasing as we leave $u_2 = -2$ on the right, and therefore $w(-2) = 5$ is a relative *minimum* value. Next:

$$\left.\frac{dw}{du}\right|_{u=0} = 8 > 0 \ \text{ and } \ \left.\frac{dw}{du}\right|_{u=1} = \frac{36}{81} - 1 < 0$$

so w is increasing as we approach $u_1 = 0.4$ from the left and w is decreasing as we leave $u_1 = 0.4$ on the right, and therefore $w(0.4) = 0.2$ is a relative *maximum* value.

3. Use the ***second derivative test*** to classify the critical values that you found in 1a. and 1.b. as relative minimum values, relative maximum values or neither.

1a. $f''(x) = 6$, so $f''(2/3) = 6 > 0$ and $f(2/3) = 2/3$ is a relative *minimum* value.

1b. $g''(t) = 12t - 18$, so $g''(4) = 30 > 0$ which means that $g(4) = -105$ is a local minimum value. Likewise, $g''(-1) = -30 < 0$, so $g(-1) = 20$ is a local maximum value.

4. Find the absolute maximum and minimum values of function $f(x) = 2x^3 - 3x^2 - 12x + 11$ on the interval $[0, 10]$. Justify your claim.

This is a *closed interval* optimization problem: the maximum and minimum values occur at critical points of the function and/or at the endpoints of the interval. So, all we need to do here is find the critical point(s) of $f(x)$ in $[0, 10]$ and then evaluate $f(x)$ at these points and at the endpoints—the biggest value we see is the maximum and the smallest value we see is the minimum.

Critical point(s):

$$f'(x) = 0 \implies 6x^2 - 6x - 12 = 0 \implies 6(x-2)(x+1) = 0 \implies \begin{cases} x_1 &= -1 \\ x_2 &= 2 \end{cases}$$

Only $x_2 = 2$ is in the interval $[0, 10]$, so we evaluate $f(x)$ at $x = 0$, $x = 2$ and $x = 10$:

$$f(0) = 11, \quad f(2) = -9 \leftarrow \textbf{\textit{minimum}} \quad \text{and} \quad f(10) = 1591 \leftarrow \textbf{\textit{maximum}}$$

5. Find the *absolute minimum* value of the function $c = 0.1q + 15 + \dfrac{100}{q}$ in the interval $(0, \infty)$. Justify your claim.

First, critical point(s):

$$\frac{dc}{dq} = 0 \implies 0.1 - \frac{100}{q^2} = 0 \implies 0.1q^2 = 100 \implies q^2 = 1000 \implies q = \pm\sqrt{1000}$$

Of the two critical points, only $q^* = \sqrt{1000}$ (≈ 31.623) lies in the interval $(0, \infty)$. I.e., there is only one critical point in $(0, \infty)$.

Next, choosing a point between 0 and $\sqrt{1000}$, we have

$$\left.\frac{dc}{dq}\right|_{q=1} = -99.9 < 0$$

and choosing a point between $\sqrt{1000}$ and ∞, we have

$$\left.\frac{dc}{dq}\right|_{q=100} = 0.09 > 0.$$

Taken together, this means that c is decreasing in the interval $(0, \sqrt{1000})$ and increasing in the interval $(\sqrt{1000}, \infty)$, which implies that

$$c(\sqrt{1000}) \approx 21.325$$

is the absolute minimum value in $(0, \infty)$.

6. Consider the function $v = u^2 e^{-5u}$.

Before addressing the four questions below, I'll find the critical points of v:

$$\frac{dv}{du} = 2ue^{-5u} - 5u^2e^{-5u} = ue^{-5u}(2 - 5u)$$

so

$$\frac{dv}{du} = 0 \implies u = 0 \;\text{ or }\; u = \frac{2}{5}$$

a. Does v have an absolute *maximum* value in the interval $(0, \infty)$? If so, find it and justify your claim. If not, explain why not.

Yes. The point $u = 0$ is not in the *open* interval $(0, \infty)$, so there is only one critical point there: $u^* = 2/5$. Testing the first derivative on either side of u^*, we find that

$$\left.\frac{dv}{du}\right|_{u=0.1} = 0.1e^{-0.5} > 0 \;\text{ and }\; \left.\frac{dv}{du}\right|_{u=1} = -3e^{-5} < 0$$

so v is increasing from 0 to 2/5 and decreasing from 2/5 to ∞, and therefore

$$v(2/5) = \frac{4}{25}e^{-2} \approx 0.22$$

is the absolute maximum value in $(0, \infty)$.

b. Does v have an absolute *minimum* value in the interval $(0, \infty)$? If so, find it and justify your claim. If not, explain why not.

No. If $0 < \tilde{u} \leq 2/5$, then $v(\tilde{u}) > v(u)$ for any $0 < u < \tilde{u}$, because v is *increasing* between 0 and 2/5. This means that $v(\tilde{u})$ is *not* the absolute minimum value of v. Likewise, if $2/5 < \tilde{u} < \infty$, then $v(\tilde{u}) > v(u)$ for any $\tilde{u} < u < \infty$, because v is *decreasing* between 2/5 and ∞, and again, this means that $v(\tilde{u})$ is *not* the absolute minimum value of v. In conclusion, there is no point $\tilde{u}$ in $(0, \infty)$ such that $v(\tilde{u})$ is minimum value.

c. Does v have an absolute *maximum* value in the interval $(-\infty, \infty)$? If so, find it and justify your claim. If not, explain why not.

No. First of all,

$$v(-1) = e^5 > \frac{4}{25}e^{-2} = v(2/5)$$

so $v(2/5)$ is not the absolute maximum value in $(-\infty, \infty)$. Additionally, since

$$v(2/5) \geq v(u)$$

for all $u \geq 0$ (see part a.), it follows that if $u \geq 0$, then $v(u)$ cannot be the absolute maximum value of v.

Next, if $u < 0$, then

$$\frac{dv}{du} = ue^{-5u}(2 - 5u) = ue^{-5u}(2 + 5|u|) < 0$$

(because $u < 0$, $e^{-5u} > 0$ and $(2 + 5|u|) > 0$) which means that v is *decreasing* in $(-\infty, 0)$. This means that given any $\tilde{u} < 0$, if $u < \tilde{u}$, then $v(u) > v(\tilde{u})$, so $v(\tilde{u})$ is *not* the maximum value of v.

It follows that there is no $\tilde{u}$ such that $v(\tilde{u}) \geq v(u)$ for all u in $(-\infty, \infty)$, so v has no absolute maximum in $(-\infty, \infty)$.

d. Does v have an absolute *minimum* value in the interval $(-\infty, \infty)$? If so, find it and justify your claim. If not, explain why not.

Yes. (No derivatives required)... $v(0) = 0$, but if $u \neq 0$, then

$$v(u) = u^2 e^{-5u} > 0 = v(0)$$

because $u^2 > 0$ and $e^{-5u} > 0$ and the product of positive numbers is positive. I.e., $v(0) = 0$ is the absolute minimum value of v in $(-\infty, \infty)$.