AMS 11A **Study Guide 5** ECON 11A

Applied optimization and curve sketching

1. Sketch a graph of the function

$$g(x) = \frac{2x+3}{x^2+2}.$$

Your sketch should clearly indicate: x and y intercepts; critical points and relative extreme values; horizontal asymptotes (i.e.,limits at $\pm\infty$); intervals where the function is increasing or decreasing; intervals where the graph is concave up or down; inflection points.

Hint: to find the intervals where $g(x)$ is concave up and where it is concave down, it will be helpful to know that the roots of the equation $2x^3 + 9x^2 - 12x - 6 = 0$ are (approximately) $r_1 \approx -5.49$, $r_2 \approx -0.4$ and $r_3 \approx 1.38$.

Solution:

(i) ***x,y-intercepts.*** The y intercept is $g(0) = \frac{3}{2}$. The x intercepts are the points where $g(x) = 0$, and in this example, there is only one, $x_0 = -\frac{3}{2}$.

Critical points: $g'(x) = \dfrac{2(x^2+2) - 2x(2x+3)}{(x^2+2)^2} = \dfrac{-2x^2 - 6x + 4}{(x^2+2)^2} = -2 \cdot \dfrac{x^2+3x-2}{(x^2+2)^2}.$

Now solve $g'(x) = 0$: Note that $g'(x) = 0$ if and only if the numerator above on the right is zero, i.e., we have to solve the quadratic equation $x^2 + 3x - 2 = 0$, for which we use the quadratic formula to find the two critical points

$$x_1 = \frac{-3-\sqrt{17}}{2} \approx -3.56 \quad \text{and} \quad x_2 = \frac{-3+\sqrt{17}}{2} \approx 0.56.$$

The corresponding y-values are $y_1 = g(x_1) \approx -0.28$ and $y_2 = g(x_2) \approx 1.78$, so the critical points on the graph are (approximately) $(-3.56, -0.28)$ and $(0.56, 1.78)$.

(ii) ***The intervals on which*** $g(x)$ ***is increasing and decreasing.*** For this we evaluate $g'(x)$ to the left of the smaller critical point, between the two critical points and to the right of the bigger critical point. We have

$$g'(-4) = -\frac{4}{289} < 0, \quad g'(0) = 4 > 0, \quad g'(1) = -1 < 0,$$

so $g(x)$ is decreasing on the interval $(-\infty, x_1)$, increasing on the interval (x_1, x_2) and decreasing again on (x_2, ∞).

(iii) ***Concavity and inflection points.*** First, find $g''(x)$ (and simplify):

$$\begin{aligned} g''(x) &= \frac{d}{dx}\left(-2 \cdot \frac{x^2+3x-2}{(x^2+2)^2}\right) \\ &= -2 \cdot \frac{(2x+3)(x^2+2)^{\cancel{2}} - 2\cancel{(x^2+2)}(2x)(x^2+3x-2)}{(x^2+2)^{\cancel{4}\,3}} \\ &= \frac{4x^3 + 18x^2 - 24x - 12}{(x^2+2)^3}. \end{aligned}$$

Determining the intervals where $g(x)$ is concave up and concave down entails finding the intervals where $g''(x)$ is positive and where it is negative. Since the denominator of $g''(x)$ is always positive, it follows that we need to determine where the numerator of $g''(x)$ is positive and negative. This is technically difficult (but not impossible) to do, since it involves finding the roots of the equation

$$2x^3 + 9x^2 - 12x - 6 = 0.$$

The roots of this equation, as provided in the hint are $x_3 \approx -5.49$, $x_4 \approx -0.4$ and $x_5 \approx 1.38$, so to determine where $g''(x)$ is positive and negative, we evaluate $g''(x)$ between consecutive pairs of roots, to the left of the smallest one and to the right of the largest one:

$$g''(-6) = -\frac{21}{13718} < 0,\ g''(-1) = \frac{26}{27} > 0,\ g''(0) = -\frac{3}{2} < 0,\ g''(2) = \frac{11}{54} > 0,$$

and it follows that $g(x)$ is *concave down* on $(-\infty, x_3)$, *concave up* on (x_3, x_4), *concave down* on (x_4, x_5) and *concave up* on (x_5, ∞). Furthermore, the points

$$(x_3, g(x_3)) \approx (-5.49, -.25), \quad (x_4, g(x_4)) \approx (-0.4, 1.02) \ \text{ and } \ (x_5, g(x_5)) \approx (1.38, 1.48)$$

are all inflection points on the graph $y = g(x)$.

(iv) ***Horizontal asymptotes.*** We have to find the limits at $\pm\infty$ of the function $g(x)$, but this is straightforward,

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{2x+3}{x^2+2} = \lim_{x\to-\infty} \frac{2x}{x^2} = 0 \quad \text{and} \quad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{2x}{x^2} = 0,$$

using the principles we learned for limits at infinity for *rational functions* (section 10.2 of the textbook).

(v) ***Pretty picture.*** Putting all of this together, you should get a graph that looks something like this:

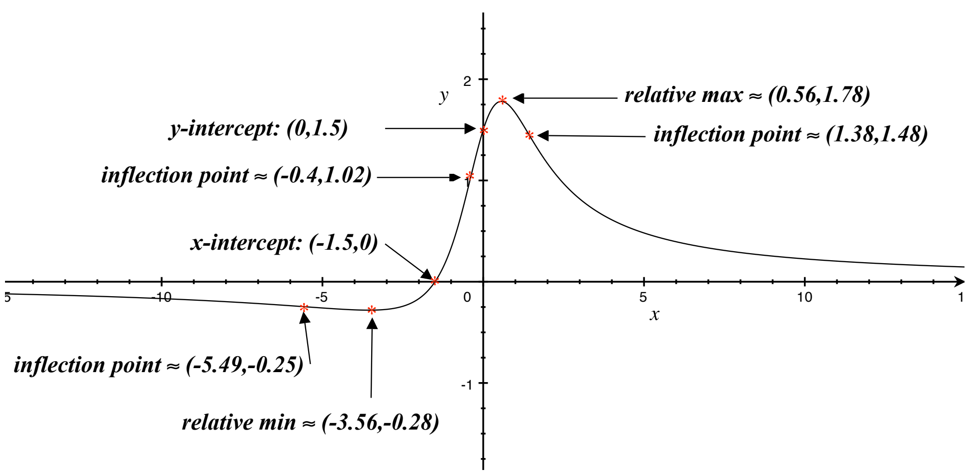

Figure 1: Graph of $y = \dfrac{2x+3}{x^2+2}$.

2. The demand equation for a monopolistic firm's product is given by $p = 830 - 2q - 0.05q^2$, where p is the price of the firm's product and q is weekly demand. The *constant* marginal cost of the firm's output is \$50 and the firm's weekly fixed cost is \$5000. Find the price the firm should set to maximize its weekly profit, as well as the corresponding output level and the max profit. Justify your claim that the price you found yields the *absolute* maximum profit.

Solution. The firm's revenue function is

$$r = pq = 830q - 2q^2 - 0.05q^3,$$

and the cost function is

$$c = 50q + 5000.$$

This means that the firm's profit function is

$$\Pi = r - c = 830q - 2q^2 - 0.05q^3 - (50q + 5000) = -0.05q^3 - 2q^2 + 780q - 5000.$$

Next we differentiate

$$\frac{d\Pi}{dq} = -0.15q^2 - 4q + 780,$$

and find the critical level(s) of demand/output for profit maximization by solving the quadratic equation

$$0.15q^2 + 4q - 780 = 0.$$

There are two roots

$$q_1 = \frac{-4 + \sqrt{16 + 468}}{0.3} = 60 \quad \text{and} \quad q_2 = \frac{-4 - \sqrt{16 + 468}}{0.3} = -\frac{260}{3}.$$

Since output must be positive, the critical output level is $q^* = 60$. The second derivative of the profit function is

$$\frac{d^2\Pi}{dq^2} = -4 - 0.3q$$

so $\Pi''(q^*) = -22 < 0$, which implies that $\Pi(q^*)$ is a relative maximum value. On the other hand, $q^* = 60$ is the ***only critical point*** in $(0, \infty)$, so

$$\Pi^* = \Pi(q^*) = \Pi(60) = 23800$$

is the *absolute* maximum profit for the firm, and the profit maximizing price is

$$p^* = 830 - 2q^* - 0.05(q^*)^2 = 530.$$

3. A firm's cost function is given by $c = 0.02q^2 + 20q + 800$. Find the level of output that minimizes the firm's *average* cost.

Solution. The average cost function is

$$\bar{c} = \frac{c}{q} = 0.02q + 20 + \frac{800}{q},$$

and the interval in this problem is $(0, \infty)$, because output q must be positive.

(a) Critical point(s):

$$\frac{d\bar{c}}{dq} = 0.02 - \frac{800}{q^2} = 0 \quad \Longrightarrow \quad q = \pm\sqrt{\frac{800}{0.02}} = \pm 200,$$

so there is only one critical point, $q^* = 200$, in the interval $(0, \infty)$.

(b) Analysis:

$$\frac{d^2\bar{c}}{dq^2} = \frac{1600}{q^3} > 0 \quad \text{for all } q > 0,$$

so $\bar{c}(200) = 28$ is the firm's absolute minimum average cost, since (i) it is a relative minimum value by the second derivative test and (ii) $q^* = 200$ is the only critical point in the interval $(0, \infty)$.

4. Farmer Jones wants to build a 4800 square foot rectangular enclosure for her vegetable garden. The enclosure will be surrounded by grade A fencing that costs \$12.00 per linear foot, and the interior of the enclosure will be subdivided into 5 equal parts using grade B fencing that costs \$8.00 per linear foot, (see Figure 1 below). What should the dimensions of the enclosure be to minimize the total cost of the fencing? What will the minimal cost be?

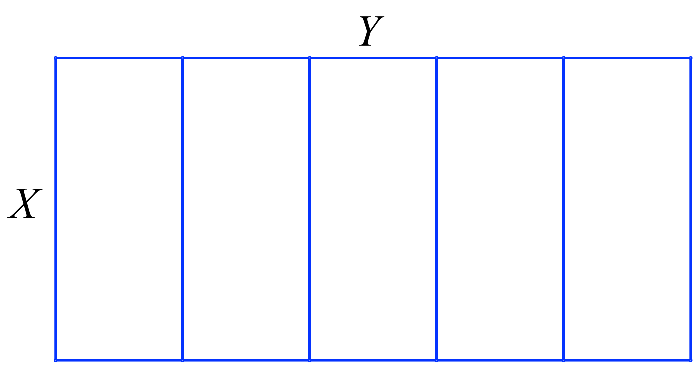

Figure 2: Farmer Jones' vegetable garden (now with the dimensions marked).

Solution. Denote by X the vertical dimension and by Y the horizontal dimension of Farmer Jones' rectangular plot, as above. Then, in terms of these dimensions and keeping in mind the different prices for grade A vs. grade B materials, the cost of the fencing is

$$C(X,Y) = \overbrace{8 \cdot (4X)}^{\text{cost of interior fencing}} + \overbrace{12 \cdot (2X + 2Y)}^{\text{cost of exterior fencing}} = 56X + 24Y.$$

Now, the area of the enclosure is $XY = 4800$, and we can use this to express Y in terms of X, $Y = 4800/X$, and substitute this into the expression above to obtain the cost as a function of X alone:

$$C(X) = 56X + \frac{115200}{X}.$$

We want to find the absolute minimum of this function on the interval $(0, \infty)$ (since lengths must be positive).

(a) Critical points:

$$\frac{dC}{dX} = 56 - \frac{115200}{X^2} = 0 \quad \Rightarrow \quad X = \pm\frac{120}{\sqrt{7}}.$$

So there is *only one critical point* in $(0,\infty)$: $X^* = \dfrac{120}{\sqrt{7}} \approx 45.356$.

(b) Analysis:

$$\frac{d^2C}{dX^2} = \frac{230400}{X^3} \quad \Rightarrow \quad \left.\frac{d^2C}{dX^2}\right|_{X=\frac{120}{\sqrt{7}}} > 0.$$

Thus, by the second derivative test, $C(120/\sqrt{7}) \approx 5079.84$ is a relative minimum value, and since $X^* = 120/\sqrt{7}$ is the only critical point in $(0,\infty)$, this is the absolute minimum cost. This minimum is attained when $X = X^* \approx 45.356$ and $Y = 4800/X^* \approx 105.83$.

(c) Conclusion: The cost minimizing dimensions of the enclosure are $X^* \approx 45.356$ and $Y \approx 105.83$, and the minimum cost to Farmer Jones is

$$C^* = C(X^*) = 56X^* + \frac{115200}{X^*} \approx 5079.84.$$

5. The *present value* of a bottle of fine cognac is given by $V(t) = 90t^{2/3}e^{-0.05t}$, where t is measured in years and $V(t)$ is measured in dollars. How many years should an investor hold the bottle of cognac before selling it, to maximize its present value? Justify your answer.

Solution. We want to find the absolute maximum value of the function $V(t)$ on the interval $(0,\infty)$ (because we can't go back in time).

(a) Critical point(s): first differentiate

$$\begin{aligned} V'(t) &= 90\left(\frac{2}{3}t^{-1/3}e^{-0.05t} - 0.05t^{2/3}e^{-0.05t}\right) && \text{product rule and chain rule} \\ &= 90e^{-0.05t}\left(\frac{2}{3}t^{-1/3} - 0.05t^{2/3}\right) && \text{factor out the exponential} \\ &= 90e^{-0.05t}\left(\frac{2}{3t^{1/3}} - 0.05t^{2/3}\right) && t^{-1/3} = \frac{1}{t^{1/3}} \\ &= 90e^{-0.05t}\left(\frac{2-0.15t}{3t^{1/3}}\right) && \text{common denominator} \end{aligned}$$

So, $V'(t) = 0$ when $2 - 0.15t = 0 \implies t^* = 2/0.15 = 40/3$ and $V'(t)$ is undefined when $t = 0$. On the other hand 0 is *not* in the interval $(0,\infty)$, so there is only the one critical point $t^* = 40/3$ to consider.

(b) Analysis and conclusion: Using the first derivative test, we see that

$$V'(1) = 90e^{-0.05}\frac{2-0.15}{3} = 55.5e^{-0.05} > 0 \quad \text{and} \quad V'(1000) = 90e^{-50}\frac{2-150}{30} = -444e^{-50} < 0$$

so

$$V^* = V(t^*) = V(40/3) \approx 259.82$$

is a relative maximum value, and since $40/3$ is the only critical point in $(0,\infty)$, V^* is the absolute maximum present value.

6. The production function for ACME Widgets is $q = 20k^{0.6}l^{0.5}$, where q is annual output, measured in 1000s of widgets, k is the capital input and l is labor input. The cost per unit of capital input is \$1000 and the cost per unit of labor input is \$5000.

a. Find the levels of capital and labor input that ***maximize*** output, given that ACME's annual production budget is $B = \$1.1$ million. *Justify your claim* that you found the *absolute maximum.*

Solution. As given, the output function is a function of two variables, k and l. We reduce the problem to a one-variable optimization problem by using the cost and budget information. I.e., we use the fact that the amount that the firm spends on production is $1000k + 5000l$, and that this must equal the firm's budget, which leads to the equation

$$1000k + 5000l = 1100000 \implies k = 1100 - 5l.$$

We substitute for k in the production function to express the output as a function of l alone:

$$q(l) = 20(1100 - 5l)^{0.6}l^{0.5}.$$

Now we have a standard optimization problem, furthermore it is a *closed interval problem* because labor and capital inputs must both be positive, so

$$0 \le l \quad \text{and} \quad 0 \le k = 1100 - 5l \implies 5l \le 1100 \implies l \le 220.$$

That is, we want to find the absolute maximum value of the function $q(l)$ on the interval $[0, 220]$.

Critical point(s): First, differentiate and simplify:

$$\begin{aligned}
q'(l) &= -60(1100 - 5l)^{-0.4}l^{0.5} + 10(1100 - 5l)^{0.6}l^{-0.5} && \text{product rule} \\
&= \frac{10(1100 - 5l)^{0.6}}{l^{0.5}} - \frac{60l^{0.5}}{(1100 - 5l)^{0.4}} && \text{use } A^{-\alpha} = \frac{1}{A^{\alpha}} \\
&= \frac{10(1100 - 5l) - 60l}{l^{0.5}(1100 - 5l)^{0.4}} && \text{common denominator} \\
&= \frac{11000 - 110l}{l^{0.5}(1100 - 5l)^{0.4}} && \text{clean up numerator.}
\end{aligned}$$

The critical points of $q(l)$ are points where either the numerator above is 0 ($q' = 0$), or the denominator is 0 (q' not defined). I.e., there are three critical points, $l_1 = 0$, $l_2 = 100$ and $l_3 = 220$. (Observe that two of the critical points are the endpoints of the interval).

Evaluate: $q(0) = 0$, $q(100) \approx 9288$ and $q(220) = 0$.

Conclusion: The firm's maximum output occurs when labor input is $l^* = 100$ and capital input is $k^* = 1100 - 5l^* = 600$.

b. What is ACME's maximum output?

Answer: $q^* = q(l^*) \approx 9288$.

c. What proportion of the total budget is spent on capital input and what proportion is spent on labor input? Do you notice anything interesting about these proportions?

Answer: To maximize output, the firm spends $1000k^*/110000 = 6/11$ of its budget on capital input and $5000l^*/1100000 = 5/11$ of its budget on labor input. These proportions are exactly equal to the ratios

$$\frac{0.6}{0.6+0.5} \quad \text{and} \quad \frac{0.5}{0.6+0.5}.$$

This is not a coincidence, as we will see in 11B.

7. A firm's production function is given by $Q = 25k^{3/5}l^{2/5}$, where Q is the firm's annual output, k is the firm's annual capital input and l is the firm's annual labor input. The cost per unit of capital input is \$1,000,000 and the cost per unit of labor input is \$50,000.

a. Find the levels of capital and labor input that the firm should use to ***minimize*** the cost of producing 10000 units. What is the the firm's minimum cost? Justify your claim that the cost you found is the absolute minimum.

Solution. This problem is similar to problem **6.**, in that we initially have a two variable function to optimize, but we have additional information that we can use to express one of the variables in terms of the other, and so transform the problem into a one variable optimization problem.

In this problem we want to *minimize* the cost function

$$c = 1000000k + 50000l,$$

where k and l are the levels of capital and labor input necessary to product $q_0 = 10000$ units. Given the production function, this means that k and l must satisfy the condtion

$$25k^{3/5}l^{2/5} = 10000$$

which means that

$$l^{2/5} = \frac{10000}{25}k^{-3/5} \implies l = 400^{5/2}k^{-3/2} = 3200000k^{-3/2}.$$

Substituting this into the cost function, leads to the one variable problem of finding the minimum value of the function

$$c(k) = 1000000k + 50000 \cdot 3200000k^{-3/2} = 1000000\left(k + 160000k^{-3/2}\right)$$

on the interval $(0, \infty)$. You should notice that this is *not* a closed interval problem, since the only restriction on the variables in this problem is $k > 0$ and $l > 0$.

Differentiate: $c'(k) = 10^6\left(1 - 240000k^{-5/2}\right)$.

Find critical point(s):

$$c'(k) = 0 \implies 1 - 240000k^{-5/2} = 0 \implies k^{5/2} = 240000 \implies k^* = 240000^{2/5} \approx 141.933.$$

The critical level of capital input is $k^* \approx 141.933$ and the critical level of labor input is $l^* = 3200000(k^*)^{-3/2} \approx 1892.445$.

Second derivative test: $c''(k) = 10^6 \cdot 600000k^{-7/2}$, so $c''(k^*) > 0$ because $k^{-7/2} > 0$ for *all* $k > 0$. It follows that $c(k^*)$ is a relative minimum, and since k^* is the only critical point in $(0, \infty)$, it follows that

$$c^* = 1000000k^* + 50000l^* \approx \$236{,}555{,}242.77$$

is the *absolute* minimum cost of producing 10000 units.

b. Find the levels of capital and labor input that the firm should use to ***minimize*** the cost of producing q units. Express the optimal input levels and the minimum cost in terms of the output q. Once again, justify your claim that you found the firm's absolute minimum cost to produce q units.

Solution. The (only) difference between this problem and part **a.** is that the target output is q instead of 10000. The solution process is exactly the same.

Solve for l in terms of k:

$$25k^{3/5}l^{2/5} = q \implies l^{2/5} = \frac{q}{25}k^{-3/2} \implies l = \frac{q^{5/2}}{3125}k^{-3/2}.$$

Express the cost in terms of k (and the parameter q):

$$c = c(k;q) = 1000000k + 50000 \cdot \frac{q^{5/2}}{3125}k^{-3/2} = 16\left(62500k + q^{5/2}k^{-3/2}\right).$$

Find critical point(s): The target output is unspecified *but fixed* — the only variable here is k, and the derivative with respect to k is

$$\frac{dc}{dk} = 16\left(62500 - 1.5q^{5/2}k^{-5/2}\right),$$

and

$$\begin{aligned}\frac{dc}{dk} = 0 &\implies 62500 = 1.5q^{5/2}k^{-5/2}\\ &\implies k^{5/2} = 0.000024q^{5/2}\\ &\implies k^*(q) = (0.000024)^{2/5}q \quad (\approx 0.0141933q).\end{aligned}$$

Second derivative test: $c'' = 60q^{5/2}k^{-7/2}$, so $c''(k^*;q) > 0$, which implies that $c(k^*;q)$ is the absolute minimum cost, since it is a relative minimum and there is only one critical point in $(0, \infty)$.

Conclusion: The cost-minimizing level of capital input for producing q units of output is $k^*(q) = (0.000024)^{2/5}q$, and cost-minimizing level of labor input is

$$l^*(q) = \frac{q^{5/2}}{3125}(k^*)^{-3/2} = \frac{(125000/3)^{3/5}}{3125}q \quad (\approx 0.1892445q).$$

Finally, the *minimum cost* to produce q units is

$$c^*(q) = 1000000k^*(q) + 50000l^*(q) \approx (14193.3 + 9462.22)q = 23655.52q.$$