Continuous Functions, Smooth Functions and the Derivative

1.   Continuous functions

One of the things that economists like to do with mathematical models is to extrapolate the general behavior of an economic system, e.g., forecast future trends, from theoretical assumptions about the variables and functional relations involved. In other words, they would like to be able to tell what's going to happen next, based on what just happened; they would like to be able to make long term predictions; and they would like to be able to locate the extreme values of the functions they are studying, to name a few popular applications.

Because of this, it is convenient to work with *continuous* functions.

Definition 1.

(i) *The function $y = f(x)$ is* **continuous at** *the point x_0 if $f(x_0)$ is defined and*

$$(1.1) \qquad\qquad \lim_{x \to x_0} f(x) = f(x_0).$$

(ii) *The function $f(x)$ is* **continuous in** *the interval $(a, b) = \{x | a < x < b\}$, if $f(x)$ is* continuous *at every point in that interval.*

Less formally, a function is continuous in the interval (a, b) if the graph of that function is a *continuous* (i.e., unbroken) line over that interval.

Continuous functions are preferable to discontinuous functions for modeling economic behavior because continuous functions are more predictable. Imagine a function modeling the price of a stock over time. If that function were discontinuous at the point t_0, then it would be impossible to use our knowledge of the function up to t_0 to predict what will happen to the price of the stock after time t_0. On the other hand, if the stock price is a continuous function of time, then we know that in the immediate vicinity of t_0, the prices after t_0 should be close to the prices up to and including t_0.

As nice as the behavior of continuous functions is, compared to functions that are not continuous, it turns out that continuity by itself is not enough. Continuous functions can also display 'erratic' behavior, and sudden changes in direction. What we would really like to have are functions that change *smoothly*.

2.   Smooth curves

Consider the function

$$f(x) = \frac{11x(x^2 - x - 2)}{(x - 2)^4 + 33}.$$

It is a continuous function, defined for all real numbers x, and its graph is shown below in Figure **??**.
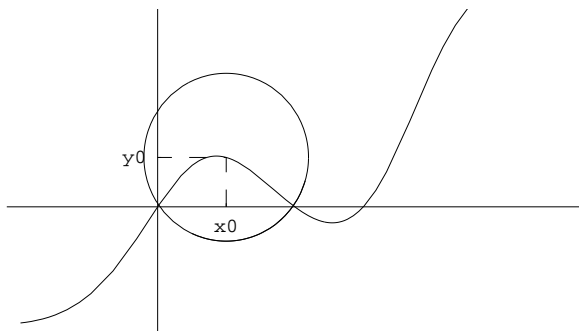
FIGURE 1. The graph of $y = f(x)$.

This graph certainly seems to be nice and smooth. But what does 'smooth' mean mathematically? Smooth usually means no rough edges or corners. To see if we can translate this into more mathematical terms, We'll focus on the point $(x_0, f(x_0)) = (x_0, y_0)$ on the graph, and zoom in.
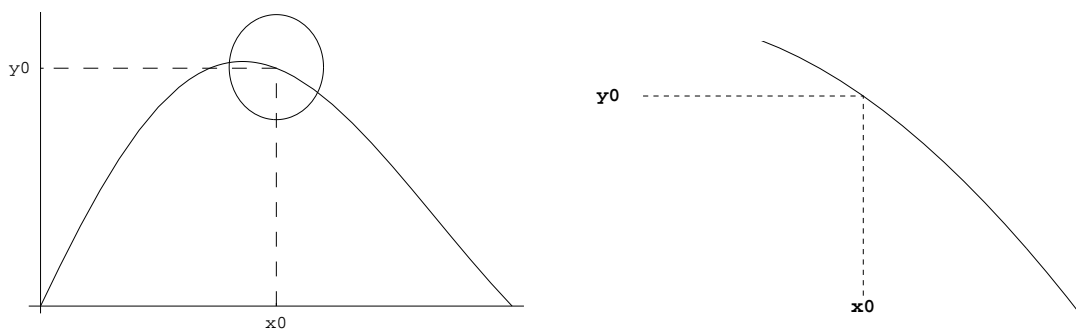

FIGURE 2. Zooming in twice on (x_0, y_0).

As you zoom in on something that is smooth, it begins to look flat (or straight) no matter how curved it was to begin with. The figure on the lefthand side of Figure ?? looks much straighter in the vicinity of the point (x_0, y_0) than in the original graph. When we zoom in again, this time on the circled portion of second graph, we get an arc that is almost straight in the immediate vicinity of (x_0, y_0), as illustrated in the graph on the righthand side of Figure ??.

For comparison's sake, the graph of $y = 1 - |x + 1|$ is displayed below in Figure ?? together with a close up. The graph on the right is a close-up of the circled portion on the graph on the left. The important thing to notice is that the corner at $(-1, 1)$ doesn't go away, it is just as sharp in the close-up as it was in the original graph.

3. Tangent lines

To capture more precisely the idea that 'smooth equals flat when you zoom in', we use the notion of a **tangent line**. Often, a tangent line is defined to as *a straight line that touches a curve at exactly one point*, (without cutting across the curve). When the curve in question is a circle then this is exactly right. But when we are dealing with more general curves, e.g., the graphs of functions, this definition doesn't work.
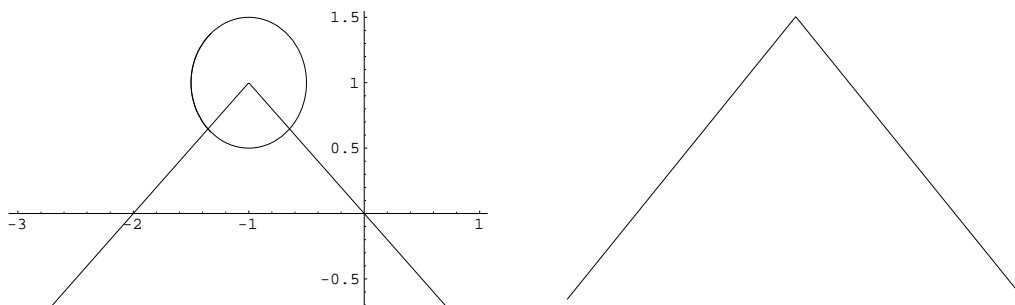
FIGURE 3. Graphs of $y = 1 - |x + 1|$, zooming in once on $(-1, 1)$.

We will need to discard the 'exactly one point' condition as well as the 'does not cut across the curve' condition. We will also need to introduce a condition that we didn't mention before, namely the condition that the tangent line be unique. *We can't have more than one tangent line to a given curve at a given point.*

A **straight line** that passes through a specific point has a direction, or slope, that characterizes it. As we zoom in on a point that lies on a smooth graph of a function, the graph through that point appears to become straight, as we saw. It seems reasonable then to associate a direction, or slope, to the graph **at that point**.

If we can find a way to compute the slope of the graph at a point on the graph, then we will have the point itself and the slope. These two pieces of information are all we need in order to write down the equation of a (unique) straight line.

Definition 2.

The tangent line to the graph of a function $y = f(x)$ at the point $(x_0, f(x_0))$ on the graph is the unique straight line that passes through that point and has the same slope as the graph at that point.

Below is the graph of the function $f(x) = (x^3 - x^2 - 2x)/((x-2)^4/11 + 3)$ that we looked at in §2, together with the tangent line to this graph at the point $(x_0, y_0))$. As you can see, this tangent line fails the old definition of 'tangency', because it both touches the graph in (at least) two points, and it cuts across the graph. But that is irrelevant now, because it **does** satisfy the new and improved Definition **??**.
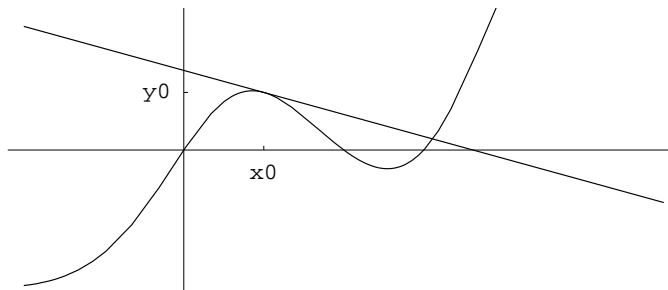

FIGURE 4. Graph of $y = f(x)$ and the tangent line at (x_0, y_0).

We can't compute the slope of the tangent line directly, at least not yet, because you usually need two points to compute a slope, and we only have one. Instead, we *approximate* it using the slope of a line, called a **secant** line, that is *near* the tangent line. The secant line intersects the graph in **two points**, the original point, (x_0, y_0), and another point of our choosing. Typically, we choose an x-value near x_0, which we call x_1. Then we compute $y_1 = f(x_1)$, which gives us the other point on the graph, namely (x_1, y_1). Now we can pass a line between the two points, and more importantly, we can compute the slope of this line! Denote this slope by m_S, and compute it as usual:
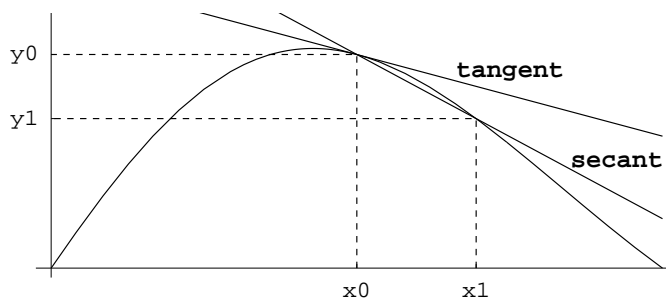
$$m_S = \frac{y_1 - y_0}{x_1 - x_0}.$$

FIGURE 5. Graph of $y = f(x)$, the tangent line and a secant line.

The tangent line and the secant line that appear in Figure **??** are relatively close, and this means that their slopes are relatively close. If we denote the slope of the tangent line by m_T, we can say that

$$(3.1) \qquad m_T \approx m_S = \frac{y_1 - y_0}{x_1 - x_0}.$$

To get better approximations we simply choose points that are closer and closer to x_0, and repeat the process. Each time, we choose an x_1 closer to x_0 than the one before and we compute $y_1 = f(x_1)$. Using formula (**??**), we get a closer approximation to m_T each time. In other words we can define the slope of the tangent line at (x_0, y_0) to be the **limit**

$$(3.2) \qquad m_T = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If this limit exists then we have the slope of the tangent line that we were seeking, and we can give a more precise definition of 'tangent line'.

Definition 3.

*The tangent line to the graph of $y = f(x)$ at the point (x_0, y_0) on the graph, is the line that passes through (x_0, y_0) with the slope m_T defined by the limit in equation (**??**).*

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4. The derivative at a point

Finding the slope of the tangent line to a graph is one of the motivations to defining the **derivative**.

> **Definition 4.**
>
> The **derivative** of the function $y = f(x)$ at the point $x = x_0$, (denoted $f'(x_0)$), is given by the limit
>
> $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \tag{4.1}$$
>
> if the limit exists. In this case we say that $f(x)$ is **differentiable** at x_0.

Note: If the limit **does not** exist then $f(x)$ is **not** differentiable at x_0.

It is very common to express the limit in equation (**??**) in a slightly different form. Instead of using the variable x, we write $x = x_0 + h$, where h is a small quantity that is approaching 0. With this in mind it follows that $x - x_0 = h$, and equation (**??**) becomes

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \tag{4.2}$$

This form of the definition of the derivative is a little easier to use in computations. In either form, we see that $f'(x_0)$ is none other than the slope of the tangent line at the point $(x_0, f(x_0))$ on the graph of the function $f(x)$.

Example: Let's find the equation of the tangent line to the graph of $f(x) = x^2 - x + 1$ at the point $(2, 3)$ on the graph. Based on our discussion thus far we know that the slope of this line is given by $f'(2)$, the derivative of $f(x)$ at 2. We compute this derivative using the definition, equation (**??**):

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{\left((2 + h)^2 - (2 + h) + 1\right) - (2^2 - 2 + 1)}{h} \\
&= \lim_{h \to 0} \frac{(4 + 4h + h^2 - 2 - h + 1) - (4 - 2 + 1)}{h} \\
&= \lim_{h \to 0} \frac{4h + h^2 - h}{h} \\
&= \lim_{h \to 0} 4 + h - 1 = 3.
\end{aligned}
$$

We want the tangent line through $(2, 3)$, and we have just computed its slope to be equal to 2, so to compute the equation of the tangent line, we use the point-slope formula:

$$y - 3 = 3(x - 2) \text{ or } y = 3x - 3.$$

Figure **??** shows the graph of $f(x)$ and the tangent line whose equation we just found.

4.1 Differentiability and continuity

Derivatives were introduced to *quantify* 'smoothness', which in turn was introduced as an improvement on continuity. It is therefore reasonable to conclude that if a function is differentiable at a point, then a fortiori, it should be continuous there. This is true, and important enough to be put in a theorem.
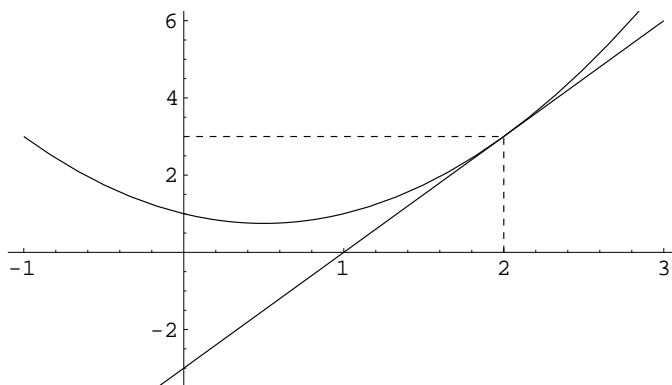
FIGURE 6. Graph of $y = x^2 - x + 1$ and the tangent line at $(2, 3)$.

Theorem 1.

If $f(x)$ is differentiable at x_0 then $f(x)$ is continuous at x_0.

Proof: Intuitively this is clear, since differentiability at x_0 means that the graph of $y = f(x)$ is smooth at the point $(x_0, f(x_0))$, which would be hard to imagine if the graph had a break at that point. Using the rules of limits, we can formally prove this intuition is correct.

Following the first version of the definition of differentiability, (**??**), if $f(x)$ is differentiable at x_0, then there is a number (the slope), $f'(x_0)$, such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

The multiplication rule for limits says that we can multiply both sides of this equation by

$$\lim_{x \to x_0} (x - x_0),$$

which gives

$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} (x - x_0) \cdot f'(x_0) = f'(x_0) \cdot \left(\lim_{x \to x_0} (x - x_0) \right) = 0,$$

because $f'(x_0)$ is a constant. Since the limit on the far left is 0, it follows that

$$\lim_{x \to x_0} f(x) = f(x_0),$$

so $f(x)$ is continuous at x_0, as claimed.

It is just as important to realize that the opposite of Theorem 1 is not true in general. In other words there are functions that are continuous but which fail to be differentiable. For example, the absolute value function $f(x) = |x|$, whose graph has a sharp corner at $x = 0$, is not differentiable at $x = 0$.

Exercise 1.

By considering the left hand limit and the right hand limit separately in the definition of the derivative, show that $f(x) = |x|$ is *not* differentiable at $x = 0$.

5.   *The derivative function*

In the previous section we defined the derivative of a function at a specific point. We also pointed out that the derivative at a point gives the slope of the tangent line to the graph of the function that passes through that point. What if we want to find the slope of the tangent line to the graph of the same function, but at a different point? One obvious possibility is to compute the limit in equation (**??**) each time for each separate point, but that becomes tedious very quickly. This process would also not be very useful for solving problems like 'find all the points on the graph where the tangent line is horizontal'.

In the limit that defines the derivative, (equation (**??**) or (**??**)), there is no particular reason to restrict ourselves to specific points like 'x_0'. We can instead try to compute the same limit with a **variable** x. If this works, we will have a formula for the derivative of the function that does not involve limits!

Definition 5.

The **derivative** of $f(x)$ is the **function** denoted by $f'(x)$ and defined for all x by the limit

$$(5.1) \qquad\qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

wherever this limit exists. If $f'(x)$ is defined for every x in an interval (a, b) then we say that $f(x)$ is **differentiable** in (a, b).

We need to emphasize that the derivative defined above is a function. The value, $f'(x_0)$, that $f'(x)$ takes at the point x_0 is the slope of the tangent line to the graph of the *original* function, $y = f(x)$, at the point on the graph $(x_0, f(x_0))$.

Comment: It follows immediately from the definitions of continuity and differentiability, and from Theorem 1 that if a function is differentiable in an interval then it is also continuous in that interval.

Example: Let's return to the previous example, and compute the derivative of $f(x) = x^2 - x = 1$ as a function of x. Using the definition (**??**) we get

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{((x+h)^2 - (x+h) + 1) - (x^2 - x + 1)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h + 1 - x^2 + x - 1}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 - h}{h} = \lim_{h \to 0} 2x + h - 1 = 2x - 1.
\end{aligned}
$$

So, the derivative of $f(x) = x^2 - x + 1$ is the **function** $f'(x) = 2x - 1$.

Comparing to the slope that we computed in §3 we find that according to our new approach we have $f'(2) = 2 \cdot 2 - 1 = 3$, which agrees with our previous result! But now we have no further limits to compute to find slopes for this function. If we want the slope of the tangent line to $f(x) = x^2 - x + 1$ at the point $(3, 7)$, we simply plug $x = 3$ into the derivative $f'(x) = 2x - 1$. This gives $f'(3) = 2 \cdot 3 - 1 = 5$, and the equation of the tangent line is given by

$$y - 7 = 5(x - 3) \text{ or } y = 5x - 8.$$

5.1 *Summary*

- Given a function $y = f(x)$, we can find its derivative, $f'(x)$, which is also a function, by computing the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

  at every point where the limit exists.
- The value, $f'(x_0)$, of the derivative at the point x_0 is equal to the slope of the graph at the point $(x_0, f(x_0))$ on the graph.
- The equation of the line tangent to the graph $y = f(x)$ at the point $(x_0, f(x_0))$ can be found using the point-slope formula, $y - f(x_0) = f'(x_0) \cdot (x - x_0)$, or

$$(5.2) \qquad\qquad y = f(x_0) + f'(x_0) \cdot (x - x_0).$$

Exercise 2.

a. Use the definition, (i.e., the limit in (**??**)), to compute the derivative of the function

$$f(x) = x^3 - x.$$

b. Find the equations of the tangent lines to the graph $y = x^3 - 3x$ at the points $(1, -2)$ and $(2, 2)$.