Example 1. What happens to the expression

$$P(x) = (1 + 0.02x)^{1/x}$$

as x approaches 1?

x	$P(x)$
1.5	~ 1.019901
1.25	~ 1.01995
1.1	~ 1.01998
1.01	~ 1.09998
0.5	1.0201
0.9	~ 1.02002
0.99	~ 1.020002

Observation:

As x approaches 1, $P(x)$ appears to be approaching $1.02 = P(1)$.

Example 2. What happens to the expression

$$P(x) = (1 + 0.02x)^{1/x}$$

as x approaches 0?

x	$P(x)$
1	1.02
1/2	1.0201
1/12	$\sim$ 1.020184356
1/52	$\sim$ 1.020197417
1/365	$\sim$ 1.020200781
1/20000	$\sim$ 1.020201329

The variable x can also approach 0 through negative values:

x	$P(x)$
-1	~ 1.020408163
$-1/2$	~ 1.020304051
$-1/10$	~ 1.020221772
$-1/80$	~ 1.020203891
$-1/200$	~ 1.020202360
$-1/10000$	~ 1.020201316

Observations:

(i) $P(x)$ is not defined when $x = 0$ (why?).

(ii) It appears that as x approaches 0, $P(x)$ approaches some number

$$\tau \approx 1.0202013\ldots.$$

Example 3. What happens to the expression

$$Q(x) = \frac{\sqrt{x} - 2}{x - 4}$$

as x approaches 0?

x	$Q(x)$
1	$1/3 \approx 0.3333$
$1/2$	~ 0.3694
$1/4$	0.4
$1/9$	$3/7 \approx 0.4286$
$1/100$	$10/21 \approx 0.4762$
$1/10000$	$19900/39999 \approx 0.4975$

Observations:

(i) As x approaches 0, $Q(x)$ appears to be approaching $0.5 = Q(0)$.

(ii) We can only approach 0 from the right in this case (why?).

Example 4. What happens to the expression

$$Q(x) = \frac{\sqrt{x} - 2}{x - 4}$$

as x approaches 4?

x	$Q(x)$
3	~ 0.26795
3.5	~ 0.25834
3.9	~ 0.25158
4.5	~ 0.24264
4.1	~ 0.24846
4.01	~ 0.24984

Observations:

(i) $Q(x)$ is not defined at $x = 4$ (why?).

(ii) As x approaches 4, $Q(x)$ appears to be approaching 0.25.

Definition of 'Limit':

The limit of ***f(x)*** as $\boldsymbol{x}$ approaches $\boldsymbol{a}$ is equal to $\boldsymbol{L}$, written

$$\lim_{x \to a} f(x) = L,$$

if we can make the difference $|f(x) - L|$ as small as we like by taking $\boldsymbol{x}$ sufficiently close to $\boldsymbol{a}$, but ***not equal*** to $\boldsymbol{a}$.

If there is ***no number L*** satisfying this condition, then the limit ***does not exist***.

The technical definition:

$$\lim_{x \to a} f(x) = L$$

if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Properties of limits.

1. If $f(x) = c$ is a constant function, then $\lim_{x \to a} f(x) = c$, for any a.

2. If n is a positive integer, then $\lim_{x \to a} x^n = a^n$, for any a.

3. If β is any real number, then $\lim_{x \to a} x^\beta = a^\beta$, for any $a > 0$.

$\star$ If the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then ...

4. $$\lim_{x \to a} (f(x) \pm g(x)) = \left(\lim_{x \to a} f(x)\right) \pm \left(\lim_{x \to a} g(x)\right)$$

5. $$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$$

Properties of limits (continued).

$\star$ If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then ...

6. $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\quad \text{provided that } \lim_{x\to a} g(x) \neq 0.$$

7. If $\lim_{x\to a} f(x) = L$ and $\lim_{u\to L} g(u) = M$, then $\lim_{x\to a} g(f(x)) = M$.

7.1 If n is a positive integer, then $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$.

7.2 If β is any real number and $\lim_{x\to a} f(x) = M > 0$, then

$$\lim_{x\to a} f(x)^\beta = \left(\lim_{x\to a} f(x)\right)^\beta = M^\beta.$$

8. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

$\Rightarrow$ *Either both limits exist and are the same, or neither limit exists.*

A special limit

$$\lim_{u\to 0}(1+u)^{1/u} = e \approx 2.7182818283459.$$

We can evaluate the limit in Example 2, using this limit and the following steps.

(1) $(1+0.02x)^{1/x} = (1+0.02x)^{\frac{1}{0.02x}\cdot(0.02)} = \left[(1+0.02x)^{1/(0.02x)}\right]^{0.02}$

(2) This means that

$$\begin{aligned}\lim_{x\to 0}(1+0.02x)^{1/x} &= \lim_{x\to 0}\left[(1+0.02x)^{1/(0.02x)}\right]^{0.02}\\ &= \left[\lim_{x\to 0}(1+0.02x)^{1/(0.02x)}\right]^{0.02}\end{aligned}$$

because of property 7.2, above.

(3) Finally, rename $0.02x = u$ and observe that if $x \to 0$, then $u \to 0$, so

$$\left[\lim_{x\to 0}(1+0.02x)^{1/(0.02x)}\right]^{0.02} = \left[\lim_{u\to 0}(1+u)^{1/u}\right]^{0.02} = e^{0.02} \approx 1.02020134.$$