

Limits at infinity of *rational functions*.

Example. Find $\lim_{x \rightarrow \infty} \frac{2x^2 + 10x + 100}{x^3 + x^2 + 1}$.

(*) The rule for quotients doesn't work here, because

$$\lim_{x \rightarrow \infty} 2x^2 + 10x + 100 = \lim_{x \rightarrow \infty} x^3 + x^2 + 1 = \infty.$$

However, the denominator is *growing faster* than the numerator (why?), which indicates that the limit is probably 0.

To see that this is true, we use a little algebra to simplify:

$$\frac{2x^2 + 10x + 100}{x^3 + x^2 + 1} = \frac{\cancel{x^2} \left(2 + \frac{10}{x} + \frac{100}{x^2} \right)}{\cancel{x^3} \left(1 + \frac{1}{x} + \frac{1}{x^3} \right)} = \frac{1}{x} \cdot \frac{2 + \frac{10}{x} + \frac{100}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^3}}$$

Therefore...

$$\begin{aligned}
\lim_{x \rightarrow \infty} \frac{2x^2 + 10x + 100}{x^3 + x^2 + 1} &= \lim_{x \rightarrow \infty} \left(\frac{x^2}{x^3} \cdot \frac{2 + \frac{10}{x} + \frac{100}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^3}} \right) \\
&= \lim_{x \rightarrow \infty} \left(\frac{1}{x} \cdot \frac{2 + \frac{10}{x} + \frac{100}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^3}} \right) \\
&= \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \lim_{x \rightarrow \infty} \frac{2 + \frac{10}{x} + \frac{100}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^3}} \\
&= 0 \cdot \frac{\lim_{x \rightarrow \infty} 2 + \frac{10}{x} + \frac{100}{x^2}}{\lim_{x \rightarrow \infty} 1 + \frac{1}{x} + \frac{1}{x^3}} \\
&= 0 \cdot \frac{2 + 0 + 0}{1 + 0 + 0} = 0 \cdot 2 = 0.
\end{aligned}$$

Observation: If U is very large and $a > 0$, then aU is very large. On the other hand, if $a < 0$, then aU is very negative, i.e., $aU < 0$ and $|aU|$ is very large.

Conclusions: If $\lim_{x \rightarrow \infty} f(x) = \infty$, then

$$\lim_{x \rightarrow \infty} af(x) = \infty$$

if $a > 0$ and

$$\lim_{x \rightarrow \infty} af(x) = -\infty$$

if $a < 0$.

Moreover, if $\lim_{x \rightarrow \infty} g(x) = a > 0$, then

$$\lim_{x \rightarrow \infty} g(x)f(x) = \infty$$

and if $\lim_{x \rightarrow \infty} g(x) = b < 0$, then

$$\lim_{x \rightarrow \infty} g(x)f(x) = -\infty$$

Example: Find $\lim_{x \rightarrow \infty} \frac{x^4 + 10x - 5}{2x^3 + x^2 + 1}$.

Simplify as before:

$$\frac{x^4 + 10x - 5}{2x^3 + x^2 + 1} = \frac{x^4 \left(1 + \frac{10}{x^3} - \frac{5}{x^4}\right)}{x^3 \left(2 + \frac{1}{x} + \frac{1}{x^3}\right)} = x \cdot \frac{1 + \frac{10}{x^3} - \frac{5}{x^4}}{2 + \frac{1}{x} + \frac{1}{x^3}}$$

Therefore

$$\lim_{x \rightarrow \infty} \frac{x^4 + 10x - 5}{2x^3 + x^2 + 1} = \lim_{x \rightarrow \infty} \left(x \cdot \frac{1 + \frac{10}{x^3} - \frac{5}{x^4}}{2 + \frac{1}{x} + \frac{1}{x^3}} \right)$$

Now, $\lim_{x \rightarrow \infty} x = \infty$ and

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{10}{x^3} - \frac{5}{x^4}}{2 + \frac{1}{x} + \frac{1}{x^3}} = \frac{\lim_{x \rightarrow \infty} 1 + \frac{10}{x^3} - \frac{5}{x^4}}{\lim_{x \rightarrow \infty} 2 + \frac{1}{x} + \frac{1}{x^3}} = \frac{1 + 0 - 0}{2 + 0 + 0} = \frac{1}{2} > 0$$

so it follows that

$$\lim_{x \rightarrow \infty} \left(x \cdot \frac{1 + \frac{10}{x^3} - \frac{5}{x^4}}{2 + \frac{1}{x} + \frac{1}{x^3}} \right) = \infty.$$

Generalizing the arithmetic:

$$\begin{aligned}\frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0} &= \frac{x^m (a_m + a_{m-1} x^{-1} + \cdots + a_0 x^{-m})}{x^n (b_n + b_{n-1} x^{-1} + \cdots + b_0 x^{-n})} \\ &= x^{m-n} \cdot \frac{a_m + a_{m-1} x^{-1} + \cdots + a_0 x^{-m}}{b_n + b_{n-1} x^{-1} + \cdots + b_0 x^{-n}}\end{aligned}$$

Observation:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{a_m + a_{m-1} x^{-1} + \cdots + a_0 x^{-m}}{b_n + b_{n-1} x^{-1} + \cdots + b_0 x^{-n}} &= \frac{\lim_{x \rightarrow \infty} a_m + a_{m-1} x^{-1} + \cdots + a_0 x^{-m}}{\lim_{x \rightarrow \infty} b_n + b_{n-1} x^{-1} + \cdots + b_0 x^{-n}} \\ &= \frac{a_m + 0 + \cdots + 0}{b_n + 0 + \cdots + 0} = \frac{a_m}{b_n}\end{aligned}$$

This allows us to formulate a rule for the limit at infinity of rational functions:

$$\lim_{x \rightarrow \infty} \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0} = \begin{cases} 0 & : m < n \\ a_n/b_n & : m = n \\ \pm\infty & : m > n \end{cases}$$

Definitions:

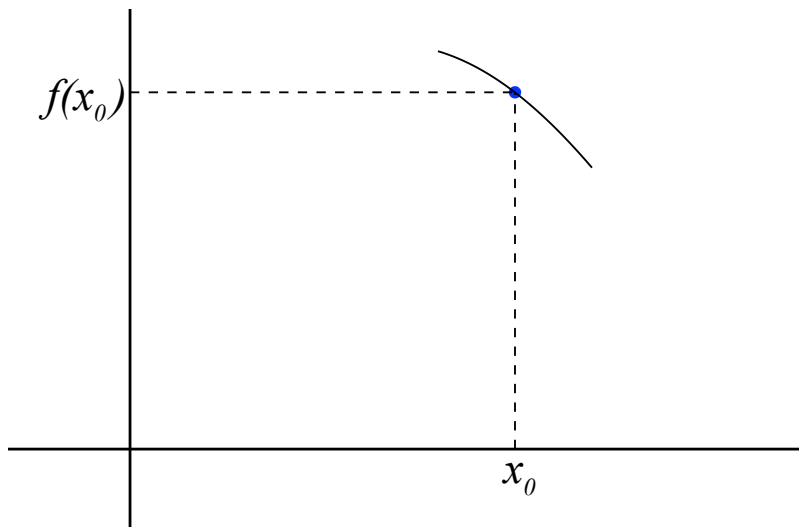
A. The function $f(x)$ is *continuous at the point* $x = x_0$ if

(i) $f(x)$ is defined at $x = x_0$, and

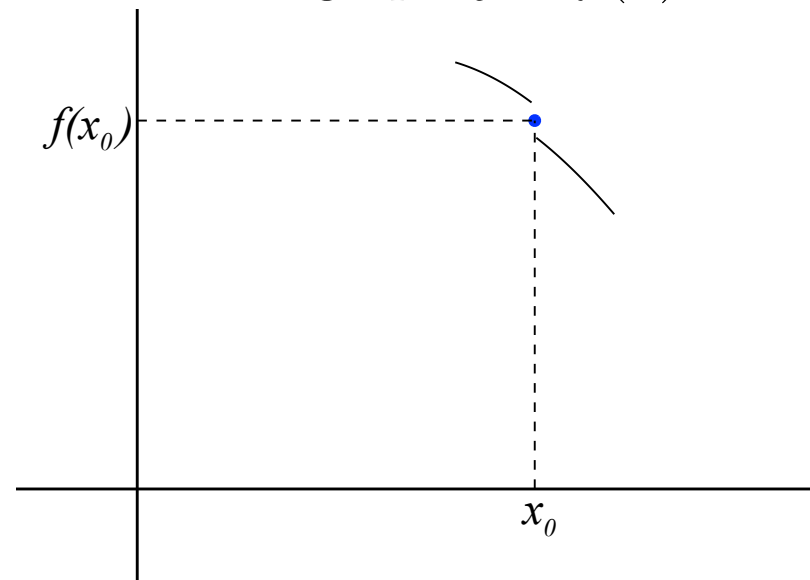
(ii) $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

(*) If $f(x)$ is *not* continuous at $x = x_0$, then we say that x_0 is a *point of discontinuity* of the function.

(*) Points of discontinuity leads to ‘breaks’ in the graph $y = f(x)$:



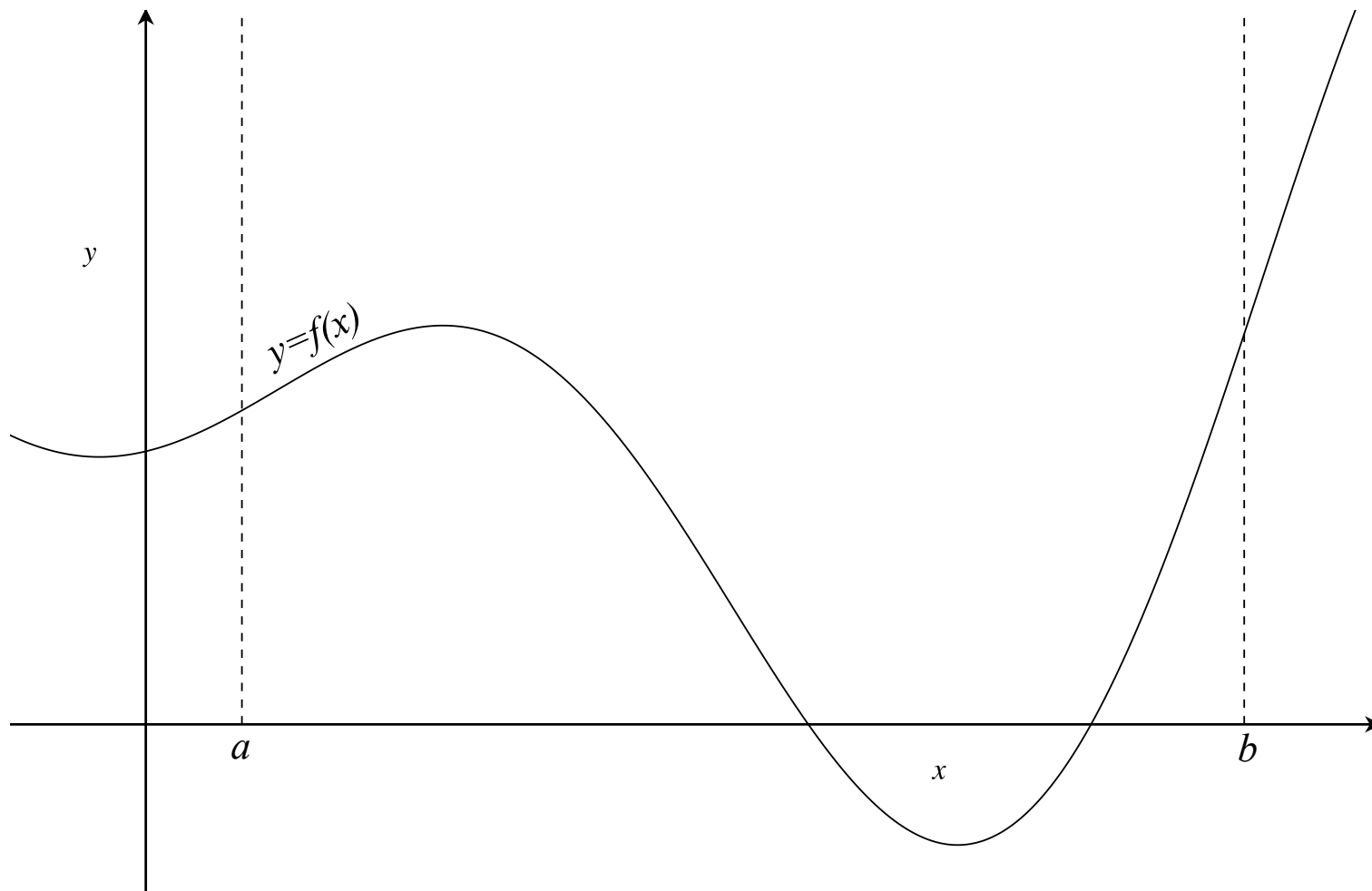
Continuous at x_0



Not continuous at x_0

B. The function $f(x)$ is *continuous in the interval* $I=(a,b)$ if $f(x)$ is continuous at every point x_0 in I .

(*) This means that the graph $y = f(x)$ can be drawn with *one continuous stroke* over the interval (a, b) .



Continuity of basic functions:

1. Constant functions are continuous on the entire real line.
2. If k is a positive integer, then $f(x) = x^k$ is continuous for all x .
3. If α is any real number, then $f(x) = x^\alpha$ is continuous for all $x > 0$.
4. The function $f(x) = e^x$ is continuous for all x .
5. The function $f(x) = \ln x$ is continuous for all $x > 0$.

Rules for combinations:

6. If $f(x)$ and $g(x)$ are both continuous at $x = x_0$, then $f(x) + g(x)$, $f(x) - g(x)$ and $f(x)g(x)$ are all continuous at x_0 . If $g(x_0) \neq 0$, then $f(x)/g(x)$ is also continuous at x_0 .

Therefore, if $f(x)$ and $g(x)$ are both continuous in the interval I , then $f(x) + g(x)$, $f(x) - g(x)$ and $f(x)g(x)$ are all continuous in I . Furthermore, $f(x)/g(x)$ is continuous at every point x in I where $g(x) \neq 0$.

7. If $g(x)$ is continuous at $x = x_0$ and $f(x)$ is continuous at $y_0 = g(x_0)$, then $f(g(x))$ is continuous at x_0 . Therefore if $g(x)$ is continuous in the interval I and $f(x)$ is continuous in

$$g(I) = \{g(x) : x \in I\}$$

then $f(g(x))$ is continuous in I .

The intermediate value property

Continuous functions have many nice properties. One of these is called the *intermediate value property*:

If $f(x)$ is continuous in the interval (a, b) and $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$) then the equation $f(x) = 0$ has a solution in the interval (a, b) .

Since there are no breaks in the graph $y = f(x)$, the graph must cross the x -axis at some point in the interval (a, b) as it travels from the point $(a, f(a))$ to the point $(b, f(b))$, as illustrated below.

