# ***The Derivative.***

**Definition:** If $y = f(x)$, then

$$y' = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

**Rules and formulas:**

**1.** If $f(x) = C$ (a constant function), then $f'(x) = 0$.

**2.** If $f(x) = x^k$ (a power function), then $f'(x) = kx^{k-1}$.

**3.** $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.

**4.** $(C \cdot f(x))' = C \cdot f'(x)$

**Notation.** Replacing $h$ by $\Delta x$ ('delta x') and $f(x+\Delta x)-f(x)$ by $\Delta y$, we can rewrite the definition of the derivative as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

And this leads to another common way of denoting the derivative:

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

That is, we use the following, interchangeable notation for the derivative:

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}.$$

Additionally, if $x_0$ is a specific point and we want to *evaluate* $dy/dx$ (or $df/dx$) at $x_0$, we write

$$\left.\frac{dy}{dx}\right|_{x=x_0} \qquad \text{or} \qquad \left.\frac{df}{dx}\right|_{x=x_0}.$$

**Linear Approximation:**

We begin with the definition of the derivative for a function $y = f(x)$ at a point $x_0$:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The expression " $\lim_{\Delta x \to 0}$ " in the equation above means that if $\Delta x$ is close to (but not equal to) 0, then

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx f'(x_0). \tag{1}$$

Multiplying this approximate equality by $\Delta x$ results in a very useful approximation:

$$f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x. \tag{2}$$

**Observation:** If $|\Delta x| < 1$, then the second approximation (2) is even more accurate than the first (1).

**Two useful variants:**

**1.** If we write $\Delta y = f(x_0 + \Delta x) - f(x_0)$, then we can express the approximation (2) as

$$\Delta y \approx f'(x_0)\Delta x. \tag{3}$$

**2.** If we write $x = x_0 + \Delta x$, so $\Delta x = x - x_0$, and add $f(x_0)$ to both sides of (2), then this approximation takes the form

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{4}$$

**Observations:**

**1.** The function of $x$ on the right of (4) is a *linear* function and this approximation formula says that if $x \approx x_0$, then we can approximate $f(x)$ by the linear function $T(x) = f(x_0) + f'(x_0)(x - x_0)$. For this reason, this approximation is frequently called ***linear approximation***.

**2.** The graph of the linear function $y = T(x)$ above is the ***tangent line*** to the graph $y = f(x)$ at the point $(x_0, f(x_0))$ (see Figure 1 below). This approximation is therefore also called *tangent line approximation*.

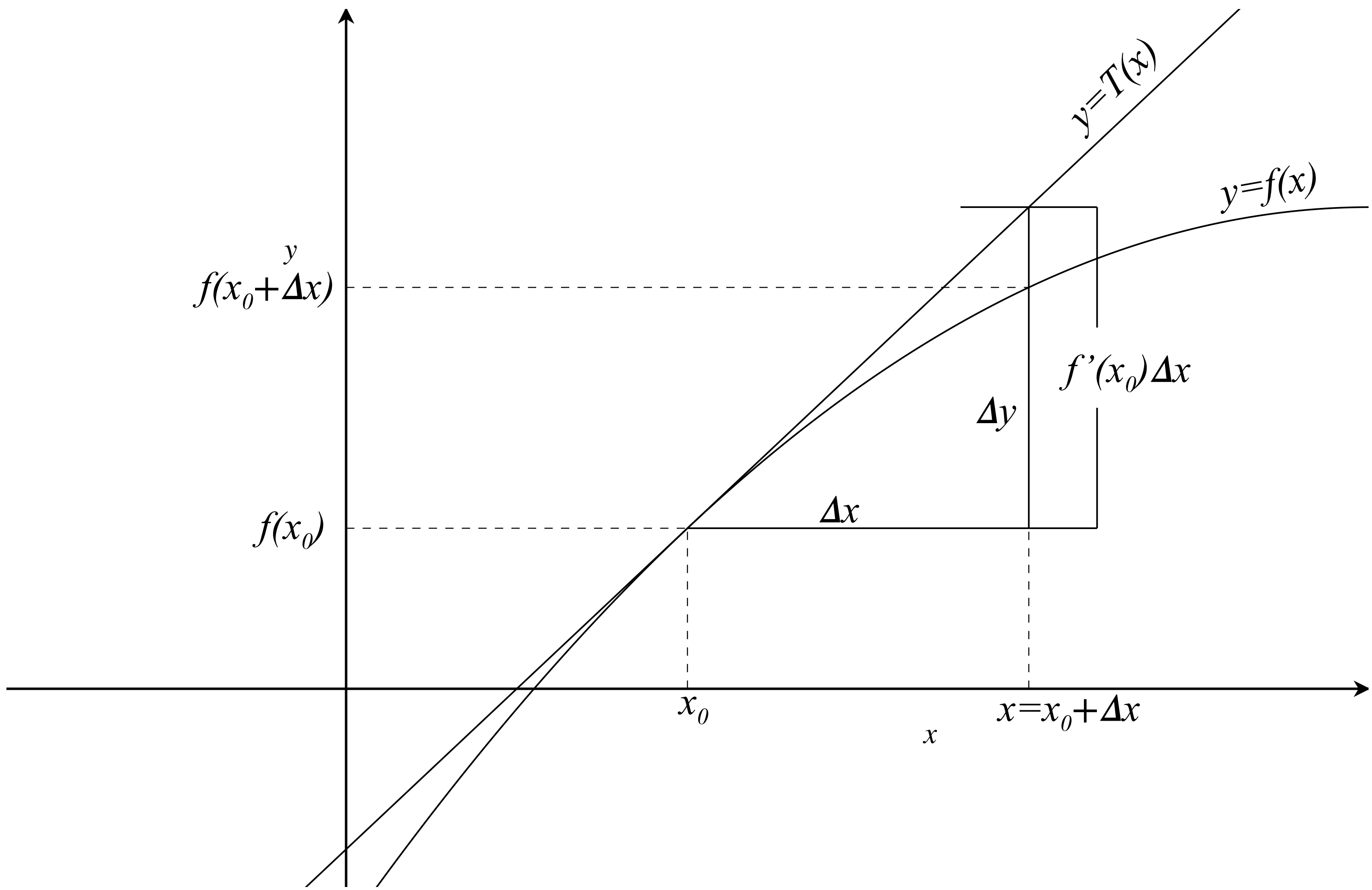


Figure 1: Linear approximation, illustrated.

**Example 1.** Find an approximate value for $\sqrt{26}$.

**(*)** Write $f(x) = \sqrt{x} = x^{1/2}$, then $f'(x) = \frac{1}{2}x^{-1/2}$.

**(*)** We don't know $\sqrt{26}$ but we do know that $\sqrt{25} = 5$, so we set $x_0 = 25$ and $x = 26$. Then, using linear approximation we have

$$\sqrt{26} = f(26) \approx f(25) + f'(25)(26 - 25) = 5 + \frac{1}{10} \cdot 1 = 5.1$$

**(*)** My calculator says that $\sqrt{26} = 5.09901951\ldots$. Based on this, the error of this approximation is less than 0.001.

**Question:** How can we obtain a more accurate estimate of $\sqrt{26}$ using linear approximation?

**Answer:** Choose $x_0$ closer to 26, so that $\Delta x = 26 - x_0$ is closer to 0. For this to be useful, we also need to know $\sqrt{x_0}$ *precisely*, which means that $\sqrt{x_0}$ must be a rational number.

For example, we can choose $x_0 = 5.1^2 = 26.01$ which satisfies both conditions. In this case,

$$f(x_0) = \sqrt{26.01} = 5.1 \quad \text{and} \quad f'(x_0) = \frac{1}{2\sqrt{26.01}} = \frac{1}{10.2} = \frac{5}{51}.$$

With this choice, linear approximation gives

$$\begin{aligned}\sqrt{26} = f(26) &\approx f(26.01) + f'(26.01)(26 - 26.01) \\ &= 5.1 + \frac{5}{51}\left(-\frac{1}{100}\right) \\ &= \frac{51}{10} - \frac{1}{1020} = \frac{5201}{1020} = 5.0990196078\ldots\end{aligned}$$

Comparing *this* estimate to the calculator value for $\sqrt{26}$ shows that the error of approximation is less than 0.0000001.

**Example 2.** The *marginal propensity to consume* of a small nation is given by

$$\frac{dC}{dY} = \frac{9Y + 10}{10Y + 1},$$

where the nation's income $Y$ and consumption $C = f(Y)$ are both measured in billions of dollars.

*The nation's current income is \$8 billion. By approximately how much will consumption increase if income increases by* \$400 *million.*

First, observe that $\Delta Y = \dfrac{400,000,000}{1,000,000,000} = 0.4$, because of the units of measurement.

Now use linear approximation in the form (3)

$$\Delta C \approx \left.\frac{dC}{dY}\right|_{Y=8} \cdot \Delta Y = \frac{9 \cdot 8 + 10}{10 \cdot 8 + 1} \cdot 0.4 \approx 0.405$$

**Interpretation:** Based on this model, if national income increases by \$400 million from its current level, national consumption will increase by about \$405 million (so the nation will incur about \$5 million in debt).