

Differentiating logarithm functions.

To differentiate $y = \ln x$, we must return to the definition...

$$\frac{d}{dx}(\ln x) = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h}$$

and simplify, using algebraic properties of $\ln x$...

$$= \lim_{h \rightarrow 0} \frac{1}{h} (\ln(x+h) - \ln x)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \ln \left(\frac{x+h}{x} \right)$$

$$= \lim_{h \rightarrow 0} \ln \left[\left(1 + \frac{h}{x} \right)^{1/h} \right]$$

and use the *continuity* of $\ln x$...

$$= \ln \left[\lim_{h \rightarrow 0} \left(1 + \frac{h}{x} \right)^{1/h} \right]$$

What next?

Remember the special limit

$$\lim_{u \rightarrow 0} (1 + u)^{1/u} = e \dots$$

and substitute (rename)

$$\frac{h}{x} = u \implies h = ux \implies \frac{1}{h} = \frac{1}{ux} = \frac{1}{u} \cdot \frac{1}{x} \dots$$

So

$$\begin{aligned} \lim_{h \rightarrow 0} \left(1 + \frac{h}{x}\right)^{1/h} &= \lim_{u \rightarrow 0} (1 + u)^{\frac{1}{u} \cdot \frac{1}{x}} \\ &= \lim_{u \rightarrow 0} \left((1 + u)^{\frac{1}{u}} \right)^{\frac{1}{x}} \\ &= \left(\lim_{u \rightarrow 0} (1 + u)^{\frac{1}{u}} \right)^{\frac{1}{x}} \\ &\quad \text{(because } f(x) = a^{1/x} \text{ is continuous)} \\ &= e^{1/x} \end{aligned}$$

Returning to $\frac{d}{dx}(\ln x)$...

$$\frac{d}{dx}(\ln x) = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h}$$

⋮

$$= \ln \left[\lim_{h \rightarrow 0} \left(1 + \frac{h}{x} \right)^{1/h} \right]$$

$$= \ln(e^{1/x})$$

$$= \frac{1}{x} .$$

I.e.,

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Example 1.

$$\frac{d}{dx} (3x^2 \ln x) = 6x \ln x + 3x^2 \cdot \frac{1}{x} = 6x \ln x + 3x$$

(product rule)

Example 2.

$$\frac{d}{dx} \ln (5x^2 + 3x + 1) = \frac{1}{5x^2 + 3x + 1} \cdot (10x + 3) = \frac{10x + 3}{5x^2 + 3x + 1}$$

(chain rule)

Example 3. Differentiate $y = \ln (5x^2)$.

We can use the chain rule again:

$$y' = \frac{1}{5x^2} \cdot 10x = \frac{10x}{5x^2} = \frac{2}{x} \dots$$

or we can simplify and then differentiate:

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$$y = \ln (5x^2) = \ln 5 + \ln x^2 = \ln 5 + 2 \ln x \implies y' = 0 + 2 \cdot \frac{1}{x} = \frac{2}{x}$$

Example 4. Find the equation of the tangent line to the graph

$$y = \ln \left(\frac{x^2 + x - 1}{4x - 3} \right)$$

at the point where $x = 1$.

Point: $y(1) = \ln \left(\frac{1 + 1 - 1}{4 - 3} \right) = \ln 1 = 0$, so the point is $(1, 0)$.

Slope: We need to compute $y'(1)$, and once again, it is easier to simplify before we differentiate:

$$y = \ln \left(\frac{x^2 + x - 1}{4x - 3} \right) = \ln(x^2 + x - 1) - \ln(4x - 3)$$

$$\implies y' = \frac{2x + 1}{x^2 + x - 1} - \frac{4}{4x - 3}$$

$$\implies y'(1) = \frac{3}{1} - \frac{4}{1} = -1.$$

So the equation of the tangent line is

$$y = y(1) + y'(1)(x - 1) \implies y = -(x - 1) \implies y = 1 - x.$$

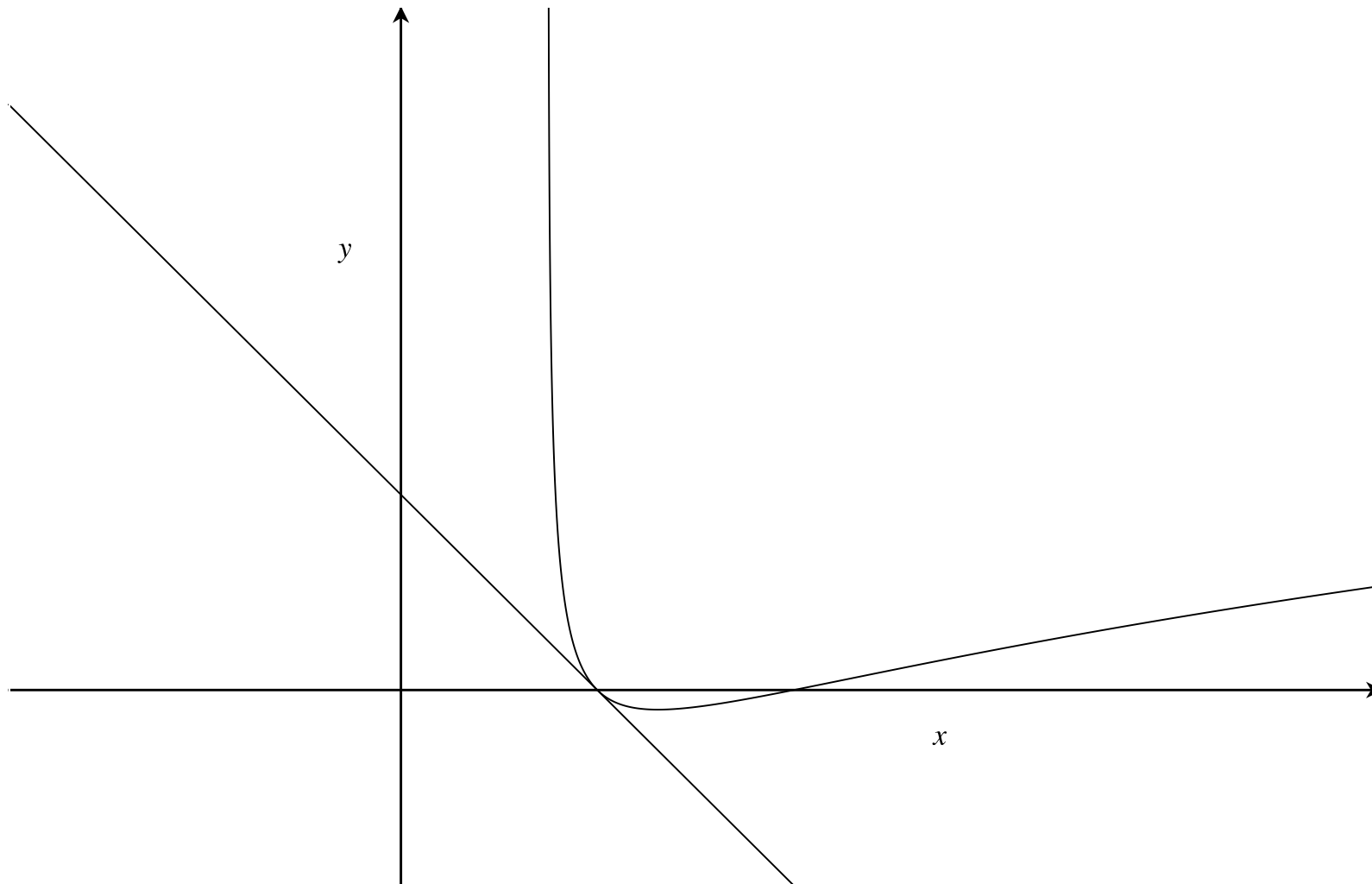


Figure 1: Graph of $y = \ln\left(\frac{x^2+x-1}{4x-3}\right)$ and its tangent line at $(1, 0)$

Logarithmic differentiation:

If $f(x)$ is any (differentiable) function, then

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

This is called the *logarithmic derivative* of $f(x)$.

In some cases, it is easier to compute the $\frac{d}{dx} \ln(f(x))$ than it is to compute $f'(x)$, because $\ln(f(x))$ is a simpler function than $f(x)$ itself.

We can take advantage of this to find $f'(x)$ because

$$f'(x) = f(x) \cdot \left(\frac{d}{dx} \ln(f(x)) \right).$$

This is called logarithmic differentiation.

We will only use this idea in one special, but **important** case...

Differentiating the exponential function.

Observation: $\ln(e^x) = x$ which is a simpler function (to differentiate) than e^x . So we will use logarithmic differentiation to find $\frac{d}{dx}e^x$:

$$\frac{d}{dx}e^x = e^x \cdot \left(\frac{d}{dx}(\ln(e^x)) \right) = e^x \cdot \left(\frac{d}{dx}x \right) = e^x \cdot 1 = e^x.$$

Example 5.

$$\frac{d}{dx} \left(3e^x x^{1/2} \right) = 3e^x x^{1/2} + 3e^x \cdot \frac{1}{2} x^{-1/2} = 3e^x x^{1/2} + \frac{3}{2} e^x x^{-1/2}$$

product rule.

Example 6.

$$\left(e^{x^2-3x+1} \right)' = e^{x^2-3x+1} (2x - 3)$$

chain rule

Example 7. Find the marginal revenue function for the firm whose demand equation is given by

$$p = 15e^{2-0.05q}.$$

First, find the revenue function

$$r = pq = 15qe^{2-0.05q}.$$

Now differentiate, using the product and chain rules:

$$\frac{dr}{dq} = 15e^{2-0.05q} + 15qe^{2-0.05q}(-0.05) = (15 - 0.75q)e^{2-0.05q}.$$

The consumption function for a small country is given by

$$C = \ln \left(\frac{e^{0.95Y}}{e^{0.2Y} + 5} \right),$$

where Y is national income, measured in \$billions.

(a) How much is consumed when $Y = 10$?

$$C(10) = \ln \left(\frac{e^{9.5}}{e^2 + 5} \right) \approx 6.983$$

(b) What is the marginal propensity to consume when $Y = 10$?

$$\begin{aligned} \frac{dC}{dY} &= \frac{d}{dY} \left(\ln \left(\frac{e^{0.95Y}}{e^{0.2Y} + 5} \right) \right) \\ &= \frac{d}{dY} (\ln (e^{0.95Y}) - \ln (e^{0.2Y} + 5)) \\ &= \frac{d}{dY} (0.95Y - \ln (e^{0.2Y} + 5)) = 0.95 - \frac{0.2e^{0.2Y}}{e^{0.2Y} + 5} \end{aligned}$$

$$\left. \frac{dC}{dY} \right|_{Y=10} = 0.95 - \frac{0.2e^2}{e^2 + 5} \approx 0.8307$$