

**The product rule.**

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

**The quotient rule:**

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

**Example 1.**  $\frac{d}{dx} ((x^2 + 4x + 5)\sqrt[3]{x}) = \dots$

$$\begin{aligned} &= \frac{d}{dx} \left( (x^2 + 4x + 5)x^{1/3} \right) = (x^2 + 4x + 5)' \cdot x^{1/3} + (x^2 + 4x + 5) \left( x^{1/3} \right)' \\ &= (2x + 4)x^{1/3} + (x^2 + 4x + 5) \cdot \frac{1}{3}x^{-2/3} \\ &= 2x^{4/3} + 4x^{1/3} + \frac{1}{3}x^{4/3} + \frac{4}{3}x^{1/3} + \frac{5}{3}x^{-2/3} \\ &= \frac{7}{3}x^{4/3} + \frac{16}{3}x^{1/3} + \frac{5}{3}x^{-2/3} \end{aligned}$$

**Check:**  $(x^2 + 4x + 5)\sqrt[3]{x} = x^{7/3} + 4x^{4/3} + 5x^{1/3}$ , so

$$\left((x^2 + 4x + 5)\sqrt[3]{x}\right)' = \left(x^{7/3} + 4x^{4/3} + 5x^{1/3}\right)' = \frac{7}{3}x^{4/3} + \frac{16}{3}x^{1/3} + \frac{5}{3}x^{-2/3}$$

**Example 2.** Find the interval(s) where the slope of  $s = \frac{3t}{t^2 + 1}$  is positive.

The slope of this graph is positive at the points  $t$  where  $ds/dt > 0$ , and...

$$\begin{aligned}\frac{ds}{dt} &= \left(\frac{3t}{t^2 + 1}\right)' = \frac{(3t)'(t^2 + 1) - 3t(t^2 + 1)'}{(t^2 + 1)^2} \\ &= \frac{3(t^2 + 1) - 3t \cdot 2t}{(t^2 + 1)^2} \\ &= \frac{3 - 3t^2}{(t^2 + 1)^2} = \frac{3(1 - t^2)}{(t^2 + 1)^2}.\end{aligned}$$

Since  $3 > 0$  and  $(t^2 + 1)^2 > 0$  for all  $t$ , it follows that  $ds/dt > 0$  when  $1 - t^2 > 0$ . I.e., the slope is positive when  $-1 < t < 1$ .

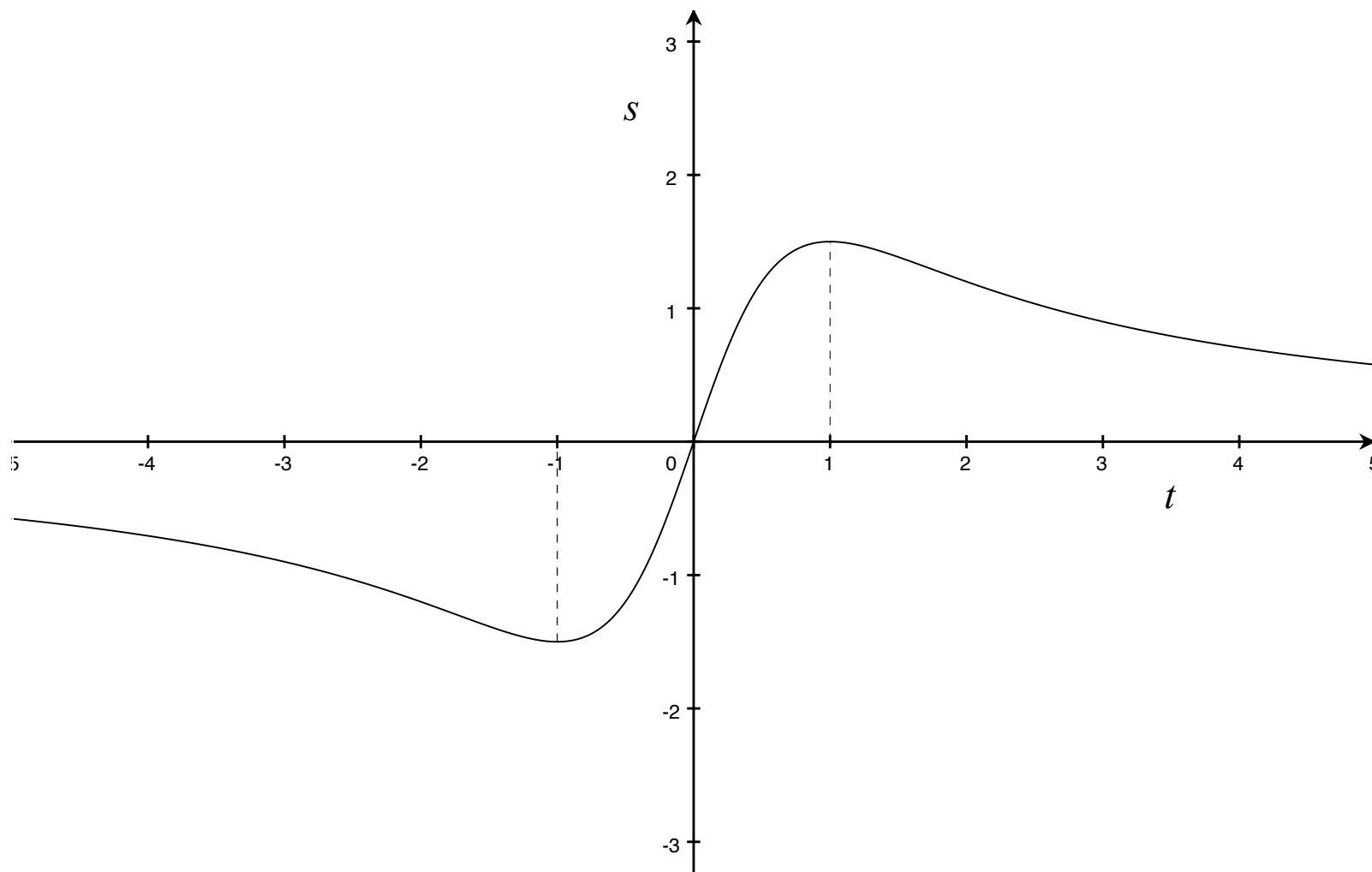


Figure 1: The graph of  $s = \frac{3t}{t^2 + 1}$ .

## The product rule.

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

Because...

$$\begin{aligned}(f(x) \cdot g(x))' &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - \overbrace{f(x)g(x+h) + f(x)g(x+h)}^{=0} - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \\ &\quad + \lim_{h \rightarrow 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h \rightarrow 0} f(x) \cdot \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \rightarrow 0} g(x+h) + f(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Because...

$$\begin{aligned} \left(\frac{f(x)}{g(x)}\right)' &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{\overbrace{f(x+h)g(x) - f(x)g(x)}^{=0} + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{f(x+h)g(x) - f(x)g(x)}{g(x)g(x+h)} \right) \\ &\quad - \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{f(x)g(x+h) - f(x)g(x)}{g(x)g(x+h)} \right) \end{aligned}$$

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{f(x+h)g(x) - f(x)g(x)}{g(x)g(x+h)} \right) \\
&\quad - \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{f(x)g(x+h) - f(x)g(x)}{g(x)g(x+h)} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \left( \frac{f(x+h)g(x) - f(x)g(x)}{h} \right) \\
&\quad - \lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \left( \frac{f(x)g(x+h) - f(x)g(x)}{h} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} \right) g(x) \\
&\quad - \lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \lim_{h \rightarrow 0} f(x) \left( \frac{g(x+h) - g(x)}{h} \right) \\
&= \frac{1}{g(x)^2} (f'(x)g(x) - f(x)g'(x)) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

**Example.** The consumption function of a small nation is given by

$$C = \frac{9Y^2 + 5Y + 100}{10Y + 1},$$

where both annual consumption  $C$  and annual income  $Y$  are measured in \$ billions.

1. Find the *marginal propensity to consume* and the *marginal propensity to save* when national income is \$8 billion.

Marginal propensity to consume: differentiate...

$$\begin{aligned} \frac{dC}{dY} &= \frac{(18Y + 5)(10Y + 1) - 10(9Y^2 + 5Y + 100)}{(10Y + 1)^2} \\ &= \frac{90Y^2 + 18Y - 995}{(10Y + 1)^2} \end{aligned}$$

Then evaluate:

$$\left. \frac{dC}{dY} \right|_{Y=8} = \frac{90 \cdot 64 + 18 \cdot 8 - 995}{81^2} \approx 0.7402$$

Marginal propensity to save...?

Use the '*national accounting identity*':

$$C + S = Y \quad \Longrightarrow \quad \frac{dC}{dY} + \frac{dS}{dY} = 1 \quad \Longrightarrow \quad \frac{dS}{dY} = 1 - \frac{dC}{dY}$$

So:

$$\left. \frac{dS}{dY} \right|_{Y=8} = 1 - \left. \frac{dC}{dY} \right|_{Y=8} \approx 1 - 0.7402 = 0.2598$$

2. What happens to the MPC as income continues to grow? What does this say about national consumption when income is large?

$$\lim_{Y \rightarrow \infty} \frac{dC}{dY} = \lim_{Y \rightarrow \infty} \frac{90Y^2 + 18Y - 995}{(10Y + 1)^2} = \lim_{Y \rightarrow \infty} \frac{90Y^2 + 18Y - 995}{100Y^2 + 20Y + 1} = 0.9$$

Interpretation: *When income is large, the nation will tend to consume about \$0.90 of each additional dollar of income.*