Observation: The derivative $f'(x)$ gives the slope (direction) of the graph $y = f(x)$ at each point $(x, f(x))$ on the graph.

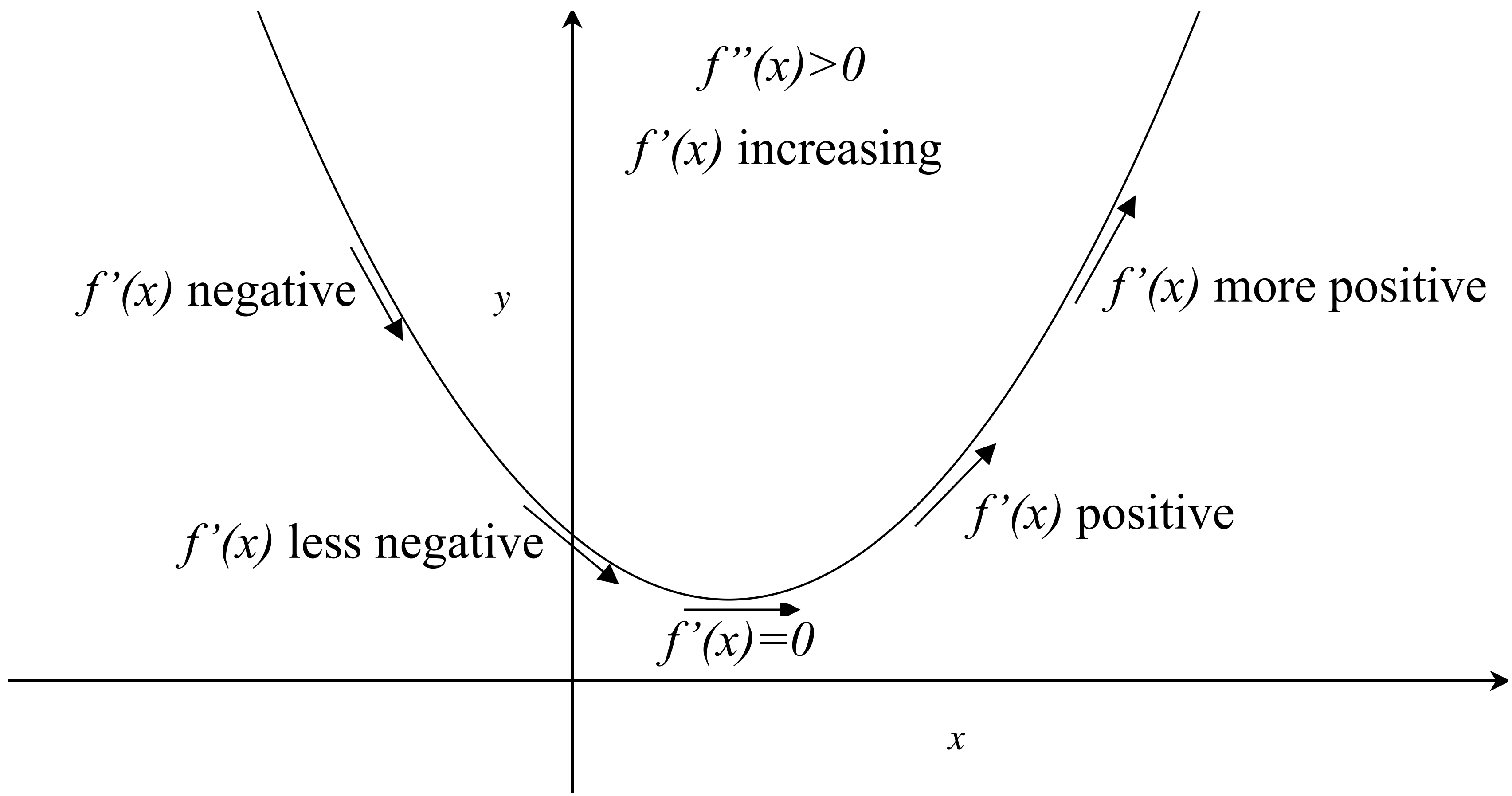
The second derivative $f''(x)$ gives the *rate of change* of the derivative $f'(x)$.

In other words, $f''(x)$ describes *how the slope of the graph is changing* — the *curvature* of the graph.

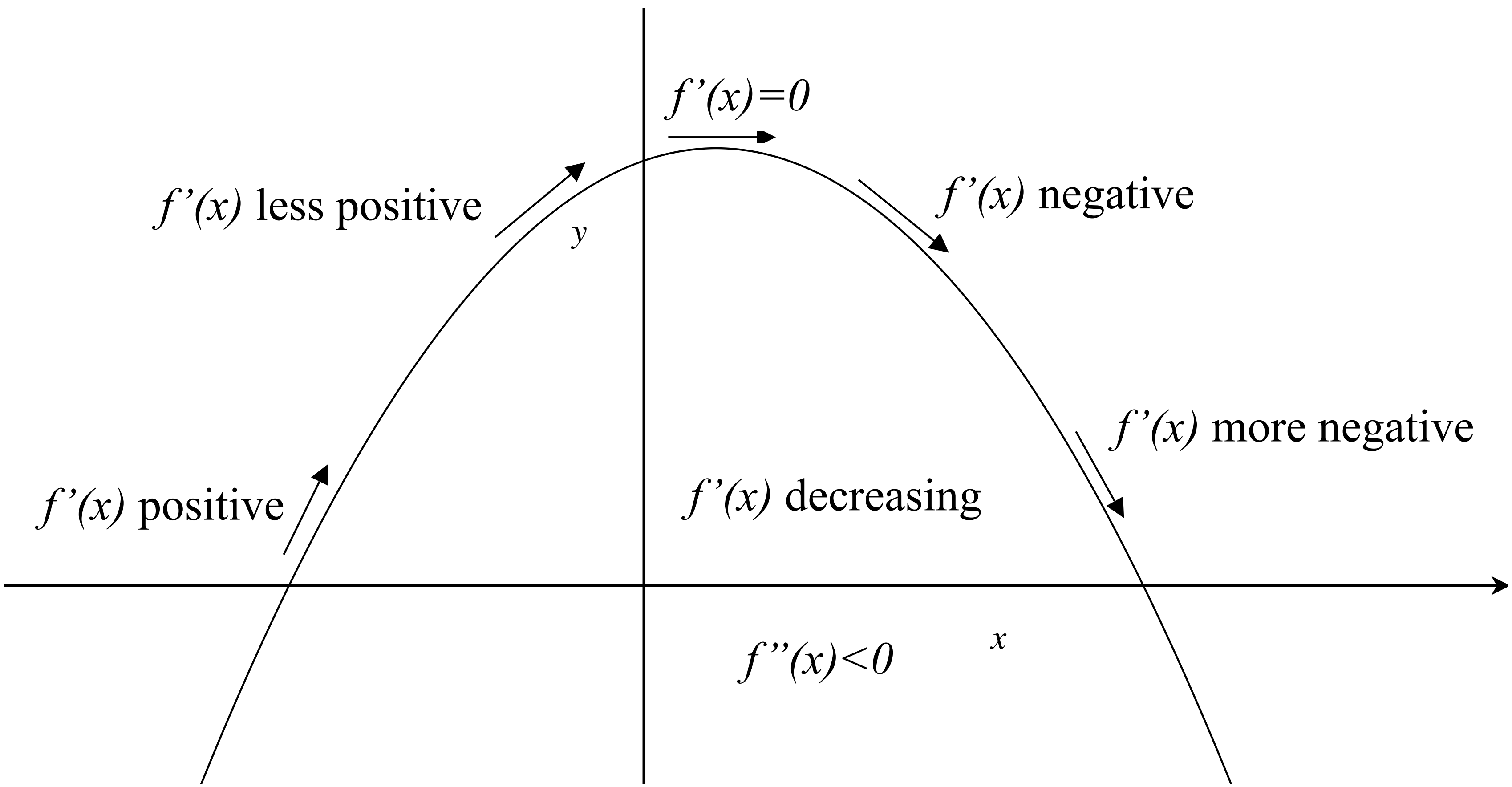
Specifically:

- If $f''(x) > 0$, then $f'(x)$ is increasing, so the slope of $y = f(x)$ is increasing and the graph is *'bending' up*.
- If $f''(x) < 0$, then $f'(x)$ is decreasing, so the slope of $y = f(x)$ is decreasing and the graph is *'bending' down*.

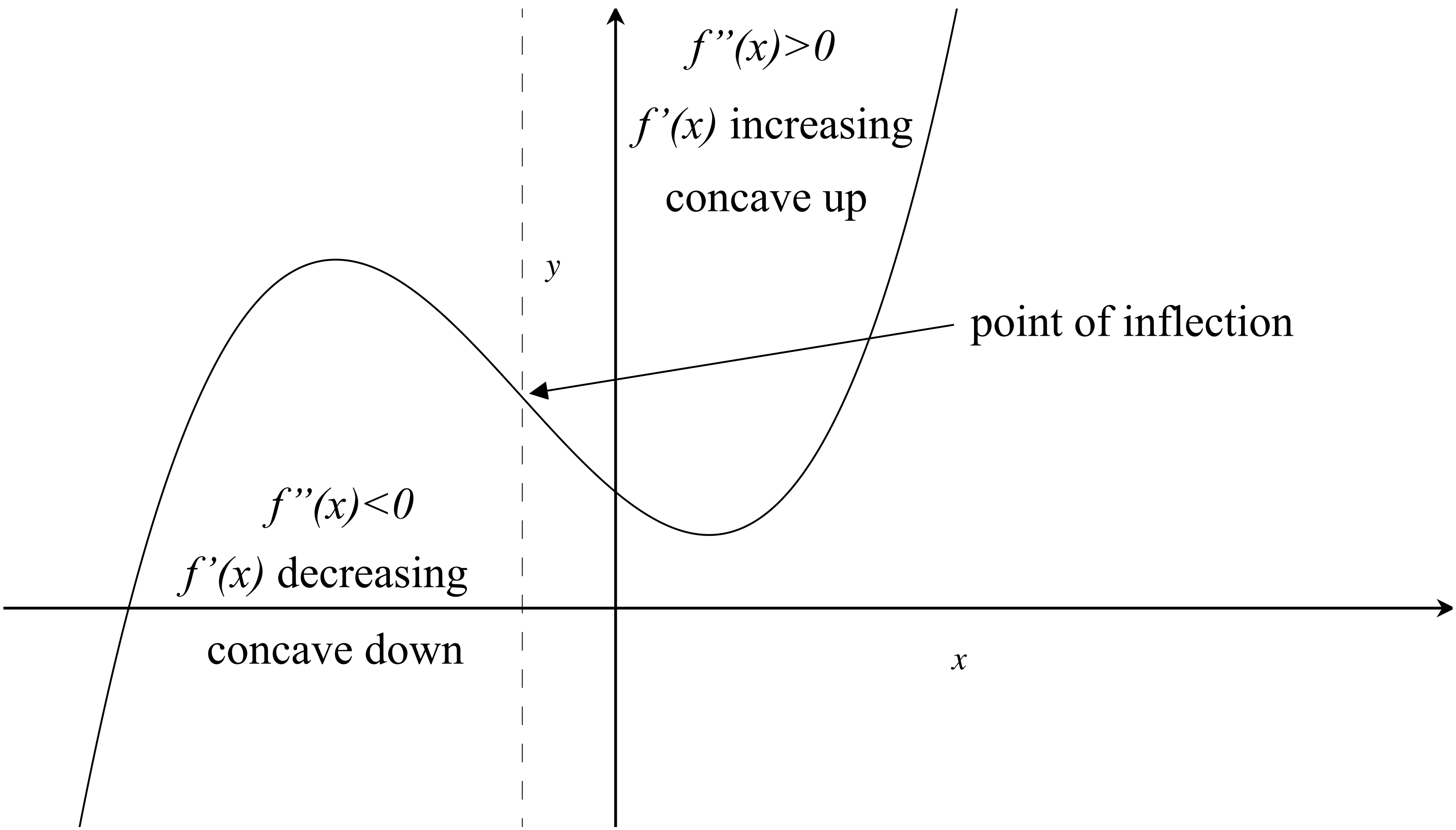
The proper term for 'bending up' is ***convex*** or ***concave up***.



The proper term for 'bending down' is ***concave*** or ***concave down.***



A point on the graph where the concavity changes is called a ***point of inflection***



Example: Find the points of inflection on the graph

$$y = \frac{1}{2}x^4 - 5x^3 + 12x^2 + 6x + 7.$$

Comment: The concavity changes when y'' changes ***sign***. This can only happen at points where $y'' = 0$ or at points where y'' is undefined.

Step 1. Find *possible* points of inflection...

$$y' = 2x^3 - 15x^2 + 24x + 6 \implies y'' = 6x^2 - 30x + 24$$

Now, y'' is defined for all x, and

$$y'' = 0 \implies 6(x-1)(x-4) = 0 \implies x = 1 \text{ or } x = 4.$$

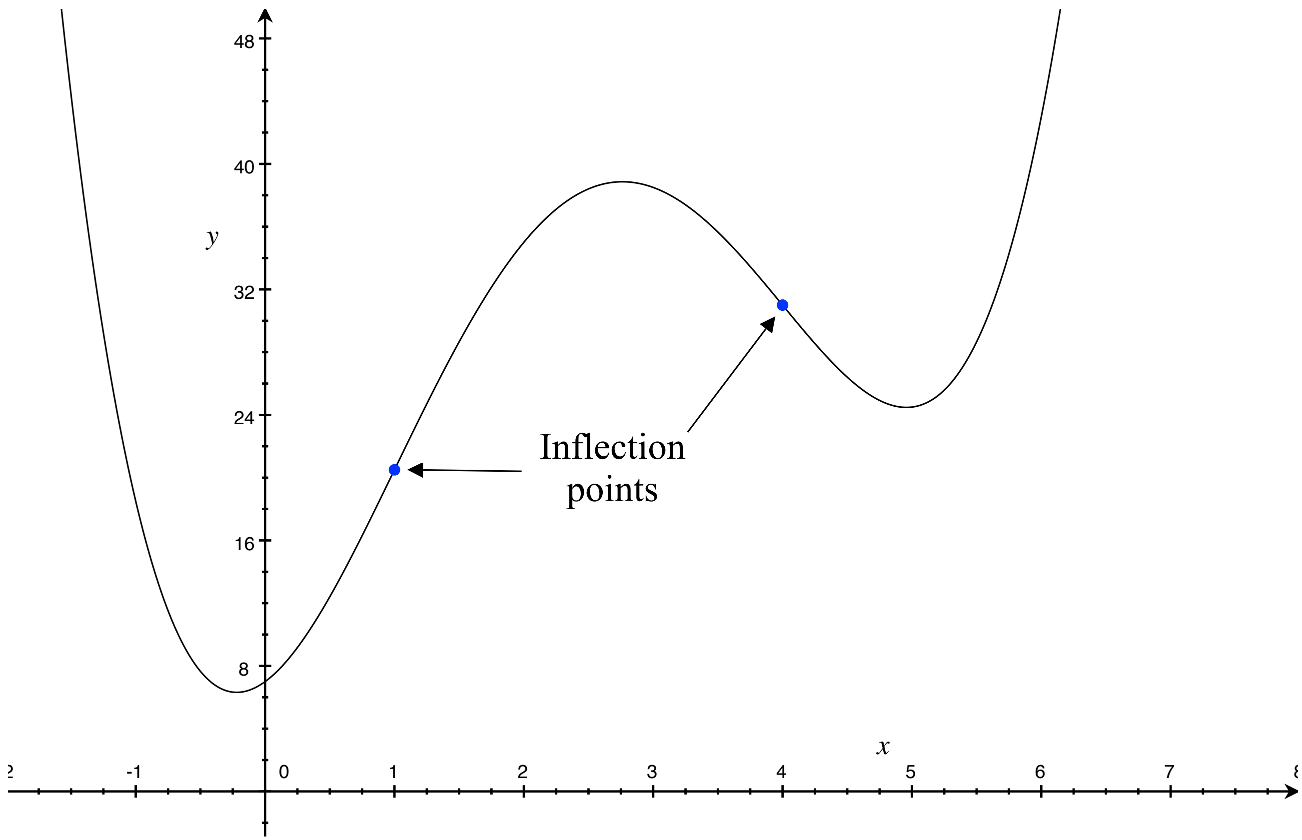
So there are *possible* points of inflection at $(1, y(1)) = (1, 20.5)$ and $(4, y(4)) = (4, 31)$.

Step 2. Analysis:

- If $x < 1$, then $y''(x) = 6(x-1)(x-4) = (+) \cdot (-) \cdot (-) = (+)$
- If $1 < x < 4$, then $y''(x) = 6(x-1)(x-4) = (+) \cdot (+) \cdot (-) = (-)$

- If $4 < x$, then $y''(x) = 6(x-1)(x-4) = (+) \cdot (+) \cdot (+) = (+)$

Conclusion: y'' does in fact change sign at both points, so both are points of inflection. The graph is concave up for $x < 1$, concave down for $1 < x < 4$ and concave up for $4 < x$.



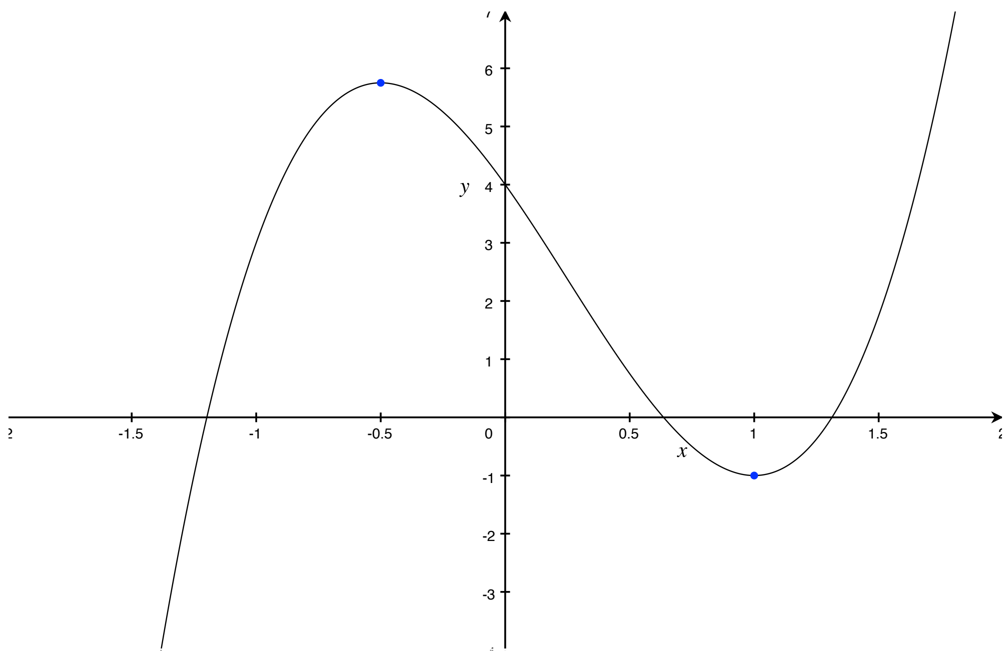
Graph of $y = \frac{1}{2}x^4 - 5x^3 + 12x^2 + 6x + 7$

Second derivative test, preview:

Consider the function $f(x) = 4x^3 - 3x^2 - 6x + 4$. To find its critical points and critical values, we set $f'(x) = 0$:

$$f'(x) = 0 \implies 12x^2 - 6x - 6 = 0 \implies 6(2x^2 - x - 1) = 0 \implies 2x^2 - x - 1 = 0.$$

Solving for x, we have $x = \frac{1 \pm \sqrt{1+8}}{4} \implies x_1 = -\frac{1}{2}$ and $x_2 = 1$, and the critical values are $f\left(-\frac{1}{2}\right) = \frac{23}{4}$ and $f(1) = -1$.



Graph of $y = 4x^3 - 3x^2 - 6x + 4$

Observation: From the graph, it is clear that $f(-1/2)$ is a relative *maximum* value and $f(1)$ is a relative minimum value.

It is also clear that the graph is *concave down* around the point $(-1/2, 23/5)$ and *concave up* around the point $(1, -1)$.

Idea: We can use the concavity of a graph around a critical point to classify the nature of the critical value:

- If $y = f(x)$ is ***concave down*** around a critical point $(a, f(a))$ on the graph, then $f(a)$ is a ***relative maximum*** value.
- If $y = f(x)$ is ***concave up*** around a critical point $(b, f(b))$ on the graph, then $f(b)$ is a ***relative minimum*** value.

This idea can be rephrased in terms of the second derivative $f''(x)$:

- If $f'(a) = 0$ and $f''(a) < 0$, then $f(a)$ is a relative maximum value.
- If $f'(a) = 0$ and $f''(a) > 0$, then $f(a)$ is a relative minimum value.

This is called the ***second derivative test***.

Example. Applying this criterion to the function

$$f(x) = 4x^3 - 3x^2 - 6x + 4,$$

we have

$$f'(x) = 12x^2 - 6x - 6 \implies f''(x) = 24x - 6.$$

At the critical point $x_1 = -1/2$

$$f''\left(-\tfrac{1}{2}\right) = -18 < 0 \implies f\left(-\tfrac{1}{2}\right) = \tfrac{23}{4} \text{ is a relative maximum value,}$$

and at the critical point $x_2 = 1$

$$f''(1) = 18 > 0 \implies f(1) = -1 \text{ is a relative minimum value,}$$

as we already observed.